Chapter 36 Diffraction

Diffraction is interference effects due to combining many waves or continuous sources and it is explained based on Huygens’ principle

1. Fresnel and Fraunhofer Diffraction
2. Diffraction from a single slit
3. Intensity in the single slit pattern
4. Multiple slits
5. The diffraction grating
6. X-Ray diffraction
7. Circular apertures and resolving power
8. Holography
Straight edge diffraction
A razor blade for example

How the shadow should look like based on geometrical optics?

What is the actual form of the shadow?

Why we do not see a sharp pattern on the shadow edges in our daily experiences?
Diffraction from a single slit

For small slits

How about large slits?
Single-slit diffraction pattern
Fresnel and Fraunhofer Diffraction

• Diffraction occurs when light passes through an aperture or around an edge.
• Diffraction is due to obstruction of the wavefront.
  – **Fraunhofer Diffraction:** when source and observer are so far away from the obstructing surface that the outgoing rays can be considered parallel or the wavefront is planar. (We will restrict our studies to the Fraunhofer Diff.).
  – **Fresnel Diffraction:** when the source or the observer is relatively close to the obstructing surface. The wavefront is no longer seen planar.
Fresnel diffraction from a single slit

(a) Imagine dividing the slit into strips. Each strip acts as a source of cylindrical secondary wavelets.

(b) Fresnel (near-field) diffraction: rays from each strip to $P$ are not parallel.
Fraunhofer diffraction from a single slit

Fraunhofer (far-field) diffraction: parallel rays (very distant screen)
Diffraction from a single slit by a monochromatic plane wave

A dark fringe occurs whenever
\[ \frac{a}{2} \sin \theta = \pm \frac{\lambda}{2} \quad \text{or} \quad \sin \theta = \pm \frac{\lambda}{a} \]

\( \pm \) indicates that there are symmetrical dark fringes above and below the point O.

In general dark fringe in single-slit diffraction happens whenever
\[ \sin \theta = \pm \frac{m\lambda}{a} \quad \text{where} \quad (m = \pm 1, \pm 2, \pm 3, \ldots) \]
Diffraction from a single slit

Location of dark fringes

\[ \sin \theta = \pm \frac{m \lambda}{a} \]

where \( m = \pm 1, \pm 2, \pm 3, \ldots \)

Typical values: \( \lambda = 500 \text{ nm} = 500 \times 10^{-9} \text{ m} \)

\( a = 0.01 \text{ mm} = 10^{-4} \text{ m} \rightarrow \left( \frac{\lambda}{a} \right) \ll 1 \)

corresponds to small angles

\[ \sin \theta \approx \tan \theta \approx \theta \rightarrow \tan \theta = \pm \frac{m \lambda}{a} ; \quad \theta \text{ is in radians} \]

\[ \tan \theta = \frac{y_m}{x} = \pm \frac{m \lambda}{a} \rightarrow \quad y_m = \pm \frac{m \lambda x}{a} ; \quad m = \pm 1, \pm 2, \ldots \]

Dark fringe location for single slit diffraction
Central fringe is a bright one
Example: Single slit diffraction

- 633 nm laser light passes through a narrow slit and the diffraction pattern is observed on a screen at 6.0m away. The distance on the screen between the centers of the first minima outside the central bright fringe is 32 mm. How wide is the slit? (0.24 mm).

- Evaluate if small angle approximation can be used?

- Find the location of the second minima on the screen (62 mm from the center).
Diffraction from a single slit

Diffraction occurs when light passes through an aperture or around an edge. When the source and the observer are so far away from the obstructing surface that the outgoing rays can be considered parallel, it is called Fraunhofer diffraction. When the source or the observer is relatively close to the obstructing surface, it is Fresnel diffraction.

Monochromatic light sent through a narrow slit of width \( a \) produces a diffraction pattern on a distant screen. Equation (36.2) gives the condition for destructive interference (a dark fringe) at a point \( P \) in the pattern at angle \( \theta \). Equation (36.7) gives the intensity in the pattern as a function of \( \theta \).

\[
\sin \theta = \frac{m\lambda}{a} \quad (m = \pm 1, \pm 2, \ldots) \quad (36.2)
\]

\[
I = I_0 \left\{ \frac{\sin[\pi a (\sin \theta)/\lambda]}{\pi a (\sin \theta)/\lambda} \right\}^2 \quad (36.7)
\]

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Intensity in the single slit Diffraction pattern

- A plane wavefront at the slit subdivided into a large number of stripes.
- The contribution of Huygens wavelets from all the stripes at a point P on a distant screen at an angle $\theta$ from the normal to the slit plane gives us the intensity distribution.
- We can use wavelet approach or superposition of the individual wavelets.
Phasors (a review)

- Each sinusoidal function is shown as a rotating vector (phasor) whose projection on the x axis shows the **instantaneous value of the function**, its length is maximum amplitude and its angle with the positive x direction is the argument of the cosine function or **phase of the wave**.

![Diagram of phasors](image.png)

\[ E(t) = E \cos(\omega t) \]
Single slit diffraction

No path difference at point $O$, Phasors are all in phase and they add up.
Intensity and Phasors

(a) Center of diffraction pattern \((\theta = 0)\): all phasors in phase

(b) Slightly off center of pattern: \(\beta = \text{total phase difference between first and last phasors}\)

Fraunhofer (far-field) diffraction: parallel rays (very distant screen)

Amplitude of the Resultant wave
Intensity in the single slit diffraction
As a function of phase difference

Amplitude of the single-slit diffraction pattern as a function of $\beta$ phase difference between the light from two ends of the slit. $E_p = 2\frac{E_0}{\beta} \sin \frac{\beta}{2} = E_0 \frac{\sin \frac{\beta}{2}}{\beta}$

Intensity proportional to square of amplitude

$I = I_0 \left[ \frac{\sin \frac{\beta}{2}}{\beta} \right]^2$

$I_0$ is intensity at the straight-ahead direction.
Intensity in the single slit diffraction as a function of geometrical parameters

$\beta$ is the phase difference

Phase difference $= \frac{2\pi}{\lambda}$ path difference $\rightarrow \beta = \frac{2\pi}{\lambda} a \sin \theta$

$I = I_o \left[ \frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}} \right]^2 = I_o \left[ \frac{\pi a \sin \theta}{\lambda} \right]^2 \rightarrow I = I_o \text{sinc}^2 \frac{\beta}{2}$

Intensity of the single slit diffraction pattern

where \[ \text{sinc} x = \frac{\sin x}{x} \]
Intensity in the single slit diffraction pattern (formula)

\[ \sin \theta = \frac{m \lambda}{a} \quad (m = \pm 1, \pm 2, \ldots) \quad (36.2) \]

\[ I = I_0 \left\{ \frac{\sin[\pi a (\sin \theta)/\lambda]}{\pi a (\sin \theta)/\lambda} \right\}^2 \quad (36.7) \]
Location of the dark fringes on the single slit diffraction pattern

Intensity of the single slit diffraction pattern: \( I = I_o \left[ \frac{\sin \left( \frac{\pi a \sin \theta}{\lambda} \right)}{(\pi a \sin \theta/\lambda)} \right]^2 \)

Location of a dark fringe: \( I = 0 \rightarrow \sin \left( \frac{\pi a \sin \theta}{\lambda} \right) = 0 \)

\( \pi a \sin \theta / \lambda = m\pi \) where \( m = 0, \pm 1, \pm 2, \pm 3, \ldots \)

a) For the case of \( m \rightarrow 0 \rightarrow I \rightarrow 0 \) is undetermined.

We use the L'Hopital's rule to determine the limit.

\[
\lim_{\sin \theta \to 0} \frac{\sin \frac{\pi a \sin \theta}{\lambda}}{\frac{\pi a \sin \theta}{\lambda}} = \lim_{\sin \theta \to 0} \frac{d}{d \sin \theta} \left( \frac{\sin \frac{\pi a \sin \theta}{\lambda}}{\frac{\pi a \sin \theta}{\lambda}} \right) = \lim_{\sin \theta \to 0} \frac{\pi a \cos \frac{\pi a \sin \theta}{\lambda}}{\lambda} = \frac{\pi a}{\lambda}
\]

So for \( \theta = 0 \rightarrow \beta = 0 \rightarrow \frac{\sin \beta/2}{\beta/2} \rightarrow 1 \) and \( I = I_o \). This is the central maximum.
Location of the dark fringes on the single slit diffraction pattern II

\[ I = I_0 \left[ \sin \left( \frac{\pi a \sin \theta}{\lambda} \right) \right]^2 \]

Location of a dark fringe:

\[ I = 0 \rightarrow \sin \frac{\pi a \sin \theta}{\lambda} = 0 \]

\[ \frac{\pi a \sin \theta}{\lambda} = m\pi \text{ where} \]

b) For the case of \( m = \pm 1, \pm 2, \pm 3, \ldots \)

**Dark fringes exist at angular locations:**

\[ \sin \theta = \frac{m\lambda}{a} \text{ with } m = \pm 1, \pm 2, \pm 3, \ldots \]

**Central maximum** occurs at \( \theta = 0 \)
Other features of the single-slit diffraction

1) Angular width of the central maxima is inversely proportional to the $\frac{a}{\lambda}$

2) When $a < \lambda$ central maximum spreads over $180^0$

Approximate intensities at the side maxima

$I \approx \frac{I_0}{(m + 1/2)^2 \pi^2}$; $m = 0, 1, 2, 3, ...$
Example 36.2: Single slit diffraction intensity

a) What is the intensity of a single-slit diffraction pattern at a point where total phase difference of the wavelets from top and bottom of the slit is 66 radian? [I = (9.2x10^{-4})I_0]

b) If this point is 7.0° away from the central maximum, how many wavelength wide is the slit? [86 lambda]
Example 36.3: Single slit diffraction

- 633 nm laser light passes through a narrow slit and the diffraction pattern is observed on a screen at 6.0m away. The distance on the screen between the centers of the first minima outside the central bright fringe is 32 mm. How wide is the slit? (0.24 mm).

- What is the intensity at a point on the screen 3.0 mm from the center of the pattern? (assume the intensity at the center is $I_0$)
Double Slit Diffraction/Interference

- Incident plane wave
- d
- a
- Single slit envelope

Double Slit Diffraction

Single slit

Double slit

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Double-slit diffraction + interference

1) The slits have almost zero width
2) The slits have finite width

Interference maxima $m_i = 0, \pm 1, \pm 2, \ldots$
Diffraction minima $m_d = \pm 1, \pm 2, \ldots$
Double Slit Diffraction/Interference

Superposition of two patterns: \( I = I_0 \cos^2 \frac{\phi}{2} \left[ \frac{\sin (\beta / 2)}{(\beta / 2)} \right]^2 \)

where \( \phi = \frac{2\pi d}{\lambda} \sin \theta \); \( \beta = \frac{2\pi a}{\lambda} \sin \theta \)
Double Slit Diffraction/Interference

Superposition of two patterns: \( I = I_0 \cos^2 \frac{\phi}{2} \left[ \frac{\sin (\beta / 2)}{(\beta / 2)} \right]^2 \)

\[ \text{Interference contribution} \quad \text{Diffraction contribution} \]

where \( \phi = \frac{2\pi d}{\lambda} \sin \theta \); \( \beta = \frac{2\pi a}{\lambda} \sin \theta \)
Three Slit Diffraction + Interference

incident plane wave

Interference be modified by slit diffraction

Single slit envelope

Note: Scale 2x that when diffraction included.
Five slit diffraction

Incident plane wave

Single slit envelope

Five Slit Diffraction
Multiple-slit Fraunhofer diffraction
Constructive interference happens when

\[ d \sin \theta = m\lambda \quad m = 0, \pm 1, \pm 2, \ldots \]
Several slits

8 slit with equal phase differences:

Minima at $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

Maxima at $0, 2\pi, 4\pi, ...$ or principal maxima.

Between the principal maxima there are N-1 minima.
Increasing number of slits

For N slits there are N-1 minima between each pair of principal maxima.

when $\phi = 2\pi m$ we have a **principal maximum**.

When $\phi = 2\pi m / N$ we have a minimum.

When $\phi$ is in between these values the waves are slightly out of phase and as a result the maxima will be narrower as the number of slits increase.

There will be small secondary maxima between the minima.
Even more slits: \( N = 16 \)

Large \( N \) corresponds to higher intensity principal maxima \( \rightarrow I_N \propto I_0 N^2 \)

The principal peaks become narrower as number of slits increase

\( \rightarrow FWHM_{\text{maxima}} \propto \frac{1}{N} \)
Example: missing maxima

• Two slits of each width a are separated a distance $d=2.5a$. Are there any missing maxima in the interference pattern? If so which are missing? ($m_i=2.5m_d$, the $5^{th}$ and $10^{th}$, etc maxima are missing)
Diffraction grating invented by Fraunhofer

- Higher number of slits results narrower principal maxima in diffraction pattern
- Maxima for different wavelengths occur at different positions
- These two properties are good for building a device that separates colors.
- We call it diffraction grating
Separation of colors with diffraction grating

\[ d \sin \theta = m \lambda \quad m = 0, \pm 1, \pm 2, \ldots \]
Diffraction grating parameters

There are two kinds of gratings: transmission & reflection grating.

Slits are called lines or rulings.

Grating spacing, \( d \), is the distance between the rulings.

Angular position of maxima for each color is given by

\[
d \sin \theta = m \lambda \quad m = 0, \pm 1, \pm 2, \ldots
\]

First order lines: \( m = \pm 1 \); Second order lines: \( m = \pm 2 \)

Overlap between the different order spectra of a grating:

\[
d \sin \theta_{1b} = \lambda_{b} \quad \theta_{1r} - \theta_{1b} = \text{angular width of the 1st order spectrum}
\]

\[
d \sin \theta_{1r} = \lambda_{r} \quad \theta_{2r} - \theta_{2b} = \text{angular width of the 2nd order spectrum}
\]

\[
d \sin \theta_{2b} = 2 \lambda_{b} \quad \theta_{2r} - \theta_{2b} = \text{angular width of the 2nd order spectrum}
\]

\[
d \sin \theta_{2r} = 2 \lambda_{r}
\]

If \( \theta_{2b} < \theta_{1r} \) then there is an overlap between the 1st and 2nd order spectra.

If \( \theta_{2b} > \theta_{1r} \) then there is no overlap between the 1st and 2nd order spectra.
Example 36.4: width of a grating spectrum

- Visible spectrum is spread between 400 & 700 nm.
  - Find the angular width of the first order visible spectrum produced by a plane grating with 600 slits per millimeter when white light falls normally on the grating ($10.9^0$).
  - Do the first and second order spectrum overlap?
  - What about second and third order spectra?
  - Does your answer depend on the grating spacing?
Grating Spectrographs

1. Light from telescope is sent along fiber-optic cables (not shown) and emerges here.

2. Light strikes concave mirror and emerges as a beam of parallel rays.

3. Light passes through diffraction grating.

4. Lenses direct diffracted light onto a second concave mirror.

5. Concave mirror reflects light to a focus.

6. An electronic detector (like the one in a digital camera) records the spectrum.
Resolution of a spectrograph
for $m^{th}$-order principal maxima: $\phi_{\text{max}} = 2m\pi$

For the minima next to it: $\phi_{\text{min}} = 2m\pi + \frac{2\pi}{N}$

We can write: $\phi_{\text{min}} = \phi_{\text{max}} + d\phi$ (d here is differential sign)

We can also write:

$$\phi = k \times \text{path difference} = \frac{2\pi}{\lambda} d \sin \theta \rightarrow d\phi = \frac{2\pi}{\lambda} d \cos \theta (d\theta) = \frac{2\pi}{N}$$

$$d \cos \theta (d\theta) = \frac{\lambda}{N}$$

For two slightly different wavelengths

$$d \sin \theta = m\lambda \rightarrow d \cos \theta d\theta = m(d\lambda)$$

$$\frac{\lambda}{N} = m(d\lambda) \rightarrow R = \frac{\lambda}{\Delta\lambda} = Nm \quad \text{Resolution}$$
Resolution of a spectrograph

- We need to distinguish (resolve) slightly differing wavelengths.

- **Chromatic resolving power** of a spectrograph: minimum wavelength difference distinguishable by the spectrograph: \( R = \frac{\lambda}{\Delta \lambda} = Nm \) where \( N \) is the number of lines and \( m \) is the order of diffraction.

- What is the resolving power of a spectrograph that can distinguish sodium doublet (589.00 and 589.59 nm)? (1000)

- What is the ruling spacing if this is done with a 10cm long grating on 1\(^{st}\) order? On 2\(^{nd}\) order? (\( m=1, \ d=0.1 \text{ mm}; \ m=2, \ d=0.2\text{ mm} \)).

- Why we can’t often use higher orders in spectroscopy although they offer higher resolution?
Summary

Multiple slit diffraction and grating

A diffraction grating consists of a large number of thin parallel slits, spaced a distance $d$ apart. The condition for maximum intensity in the interference pattern is the same as for the two-source pattern, but the maxima for the grating are very sharp and narrow. (See Example 36.4)

$$R = \frac{\lambda}{d\lambda} = Nm$$

$N$ is the number of lines

$m$ is the order of diffraction

$$I_N \propto I_0 N^2 \quad \text{and} \quad FWHM_{\text{maxima}} \propto \frac{1}{N}$$

Large $N$ provides everything we want from a grating but there is a technical limitation.
Circular aperture and resolving power
Diffraction pattern formed by a circular aperture

Angle of Airy disk: \( \sin \theta_1 = 1.22 \frac{\lambda}{D} \)

Second dark ring: \( \sin \theta_2 = 2.23 \frac{\lambda}{D} \)

Third dark ring: \( \sin \theta_3 = 3.24 \frac{\lambda}{D} \)

Tree bright rings at

\( \sin \theta = 1.63 \frac{\lambda}{D} \quad 2.68 \frac{\lambda}{D} \quad 3.70 \frac{\lambda}{D} \)

Airy disk 85% of the light falls within this disk
Diffraction pattern formed by a circular aperture

Circular aperture

\[ \sin \theta = \frac{m \lambda}{d} \]

\(d = \text{aperture diameter}\)

\[ y \approx D \frac{m \lambda}{d} \]

for maxima and minima

\[ \frac{y}{D} = \tan \theta \approx \sin \theta \approx \theta \]

for small angles \(\theta\)

Relative Intensity
0.0175

0.0042

0.00078
Resolving power of a camera

• A camera lens with focal length 50 mm and maximum aperture f/2 forms an image of an object 9.0 m away. Assume that wavelength is 500 nm.

  – if the resolution is limited by diffraction, what is the minimum distance between two points on the object that are barely resolved and what is the corresponding distance between image points?

  – how does the situation change if the lens is “stopped down” to f/16?

• Attention: not to be mistaken with chromatic resolving power of a grating
Example 36.36: diffraction grating resolving power

• The light from an iron arc includes many different wavelengths. Two of these are 587.9782 nm and 587.8002 nm. You wish to resolve these lines in first order using a grating 1.20 cm in length. What minimum number of slits per centimeter must the grating have? (R=\|/D\|=Nm=3303, N/cm=2752)
X-Ray diffraction

• X rays discovered by Wilhelm Rontgen (1845-1923)
• X rays are electromagnetic waves with λ of the order of $10^{-10}m$
• Crystalline solids have atoms arranged in a regular repeating pattern with spacing of the order of $10^{-10}m$
• Max von Laue (1879-1960) suggested crystal might act as a three-dimensional diffraction grating for x-rays
• X-rays are absorbed and re-emitted by the atoms. Each atom acts like a secondary source. The emitted waves from atoms interfere and form a diffraction pattern.
Geometry of the first x-ray diffraction experiment (1912 Van Laue and Knipping)
Scattering of x-rays from rectangular array of atoms
Scattering of x rays from array of atoms

Bragg reflection

Interference from adjacent atoms in a row. Constructive when
\[ a \cos q_a = a \cos q_r \]
This means only parallel rays or \( q_a = q_r = q \)
This is the angle with surface not the normal

Interference from adjacent atoms on two adjacent rows is constructive when
\[ 2d \sin q = m \lambda \quad (m=1, 2, \ldots) \]
Bragg condition for constructive interference
X-ray analysis of crystals

Bright maxima makes a dark point on the film whenever the Bragg Condition is satisfied.
X ray crystallography
Understanding atomic structure

• If we know the x-ray source we can learn about the crystal
• if we know the crystal we can learn about x-ray source

(a) PHYS 52 Eradat SJSU (b)
Example 36.5: X-ray diffraction

• Direct a beam of x-rays with wavelength 0.154 nm at certain planes of a silicon crystal. As you increase the angle of incidence from zero, you find the first strong maximum when the beam makes an angle of 34.5 degrees with the planes? How far apart are the planes? (d=0.136 nm)

• Will you find another maximum from these planes at another angle? (No)

• What wavelength of x-rays you need to use to get 3 maxima from the same planes? (<0.0902)
X ray diffraction

\[ 2d \sin \theta = m\lambda \quad (m = 1, 2, 3, \ldots) \quad (36.16) \]

A crystal serves as a three-dimensional diffraction grating for x rays with wavelengths of the same order of magnitude as the spacing between atoms in the crystal. For a set of crystal planes spaced a distance \( d \) apart, constructive interference occurs when the angles of incidence and scattering (measured from the crystal planes) are equal and when the Bragg condition [Eq. (36.16)] is satisfied. (See Example 36.5)

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The diffraction pattern from a circular aperture of diameter $D$ consists of a central bright spot, called the Airy disk, and a series of concentric dark and bright rings. Equation (36.17) gives the angular radius $\theta_1$ of the first dark ring, equal to the angular size of the Airy disk. Diffraction sets the ultimate limit on resolution (image sharpness) of optical instruments. According to Rayleigh’s criterion, two point objects are just barely resolved when their angular separation $\theta$ is given by Eq. (36.17). (See Example 36.6)

\[
\sin \theta_1 = 1.22 \frac{\lambda}{D} \quad (36.17)
\]
Holography
Seeing a hologram