Chapter 4
Wave Equations

Lecture Notes for Modern Optics based on
Pedrotti & Pedrotti & Pedrotti
Instructor: Nayer Eradat
Spring 2009

Wave Equation

Chapter Goal:

developing the mathematical expressions for wave motion.

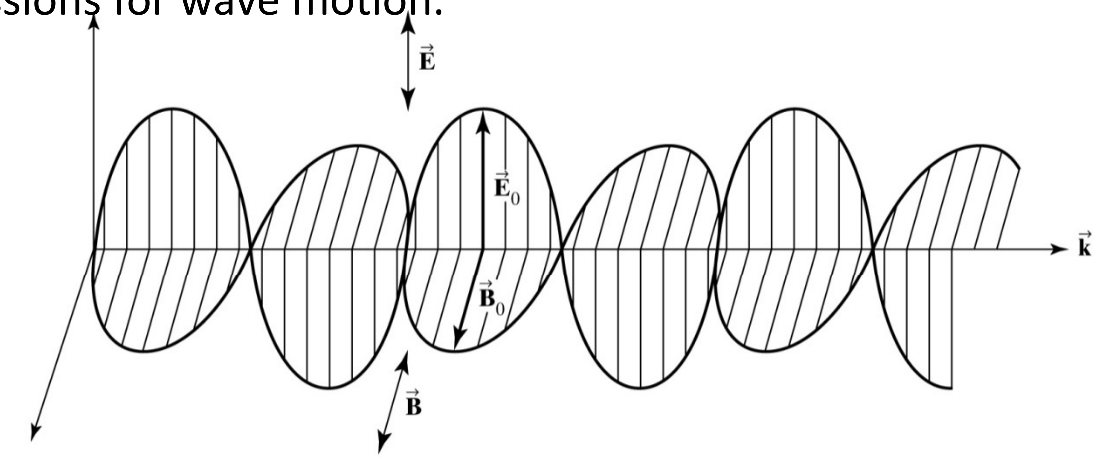
Most general case

Harmonic waves

Electromagnetic (EM) waves

Light waves

Energy delivered by such EM waves



(a)

Wave:

- A self sustaining energy-carrying <u>disturbance</u> of a medium through which it propagates.
    - Longitudinal wave: the medium is displaced in the direction of motion of the wave.
    - Transverse wave: the medium is displaced in a direction perpendicular to that of the motion of the wave.
- When a wave propagates, the disturbance advances, <u>not the medium</u>. That is why waves can propagate faster than the medium carrying them (Leonardo da Vinci).

Equation for mathematical form of a pulse

Goal: developing a mathematical expression for the most general form of a one-dimensional (1D) traveling wave. Consider:

$O(x, y) \leftarrow$ A stationary coordinate system. $O'(x', y') \leftarrow$ A moving coordinate system.

$V \leftarrow$ Relative speed of motion of $O'(x', y')$ to $O(x, y)$ in the $+x$ direction

$y' = f(x') \leftarrow$ A one dimensionnal pulse of arbitrary time-independent shape fixed to $O'(x', y')$

y or y' is the transverse displacement from equilibrium (the disturbance). As O' moves the pulse maintains its shape. Relationship between the coordinates of an arbitrary point on the pulse

$$P(x, y) \text{ and } P(x', y') \rightarrow \begin{cases} x' = x - Vt \\ y' = y = f(x') = f(x - Vt) \end{cases}$$

For a pulse moving in $-x$ direction $y' = f(x + Vt)$

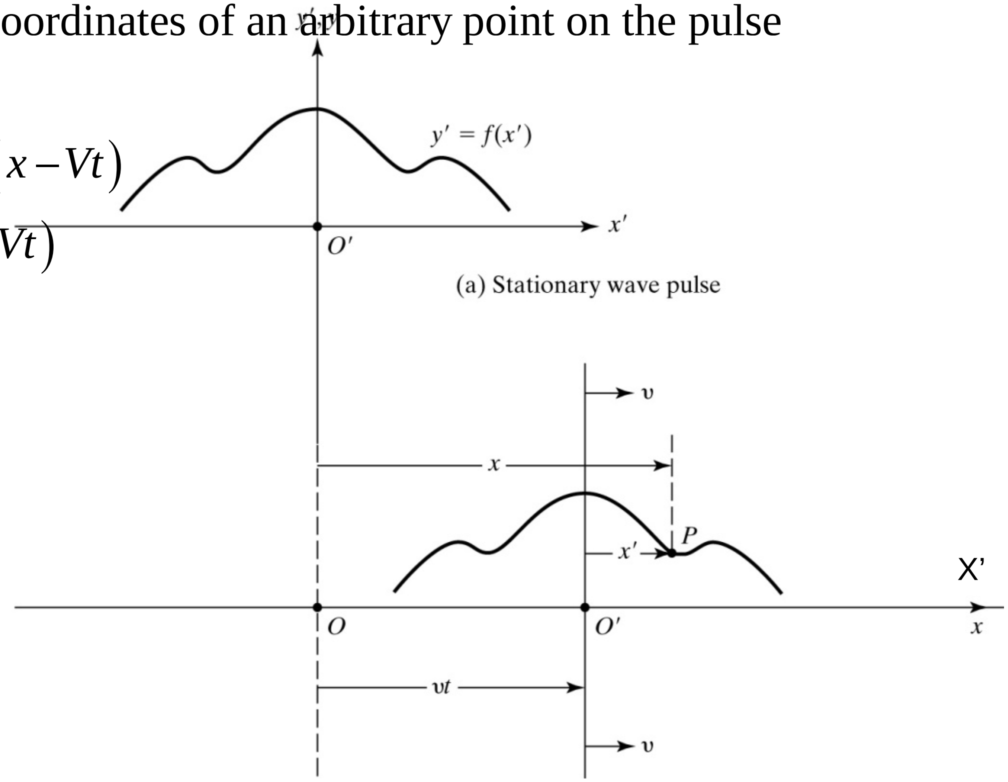
$$\underbrace{y' = f(x \mp Vt)}_{\text{Equation for mathematical form of a pulse}} \quad - \text{ for } +x \text{ and } + \text{ for } -x$$

f can be any function for example

$y = A\sin(k[x - Vt])$ traveling periodic in $+x$

$y = A(x + Vt)^2$ traveling in $-x$

$y = e^{k(x-Vt)}$ traveling in $+x$

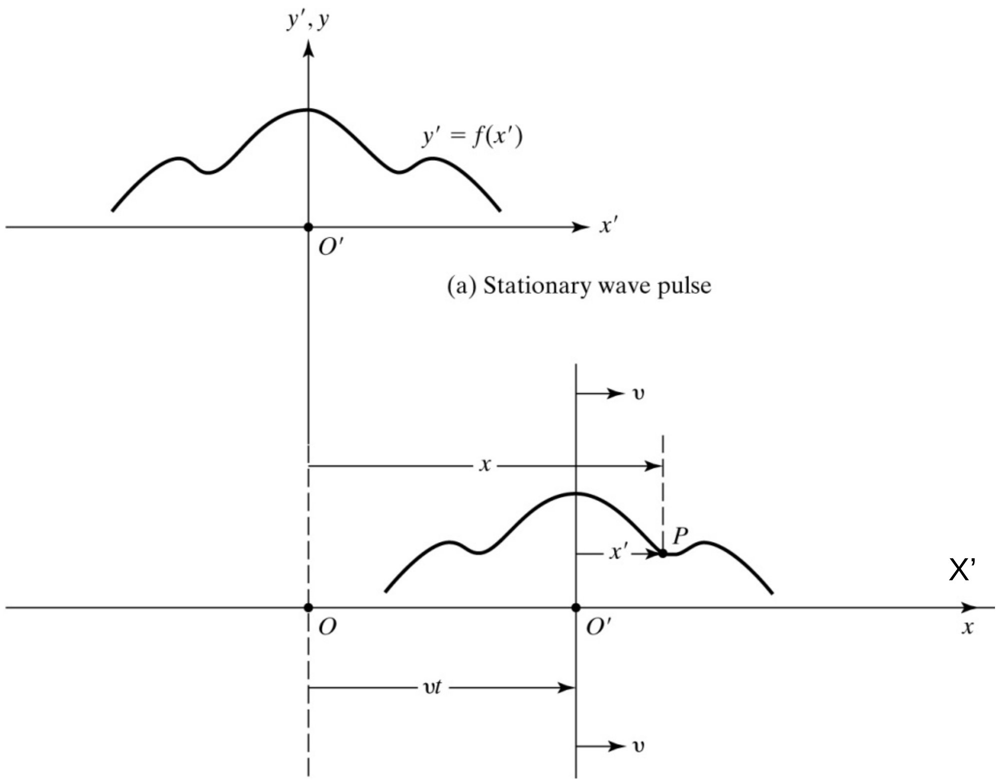


One dimensional wave equation

The wave equation is a partial differential equation that any arbitrary wave form will satisfy it.

Verify that any wave of the form $y = f\left(x \mp Vt\right)$ satisfies $\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 y}{\partial t^2}}$ the 1D wave equation.

To show that if a given function represents a traveling wave, it is sufficient to represent it in the form of $y = f\left(x \mp Vt\right)$ or prove that it satisfies the wave equation.



Harmonic waves

Harmonic waves are smooth patterns that repeat endlesly. They involve the sine and cosine functions:

$y = A\sin\left(k\left[x - Vt\right]\right)$ or $y = A\cos\left(k\left[x - Vt\right]\right)$

A and k are constants that specify a wave without altering its harmonic nature. The sin and cos functions form a complete set of functions so a linear combination of them is also a wave. Using this property we can construct complicated waveforms from linear superposition of the simple sin and cos functions.

$\sin x = \cos\left(x - \pi/2\right)$ so sin is equivalent to cosin, shifted by $\pi/2$.

Amplitude A: the maximum value of the disturbance

Wavelength λ: the repetitive spatial unit of the wave

$$\left.\begin{aligned} &A\sin k\left[\left(x+\lambda\right)+Vt\right] = A\sin k\left(x+Vt\right) \\ &\sin x = \sin\left(x+2\pi\right) \end{aligned}\right\} k\lambda = 2\pi \text{ or } k = \frac{2\pi}{\lambda}$$

Propagation constant: $\boxed{k = 2\pi/\lambda}$ related to spatial period λ

k is also called spatial frequency or number of waves in 2π

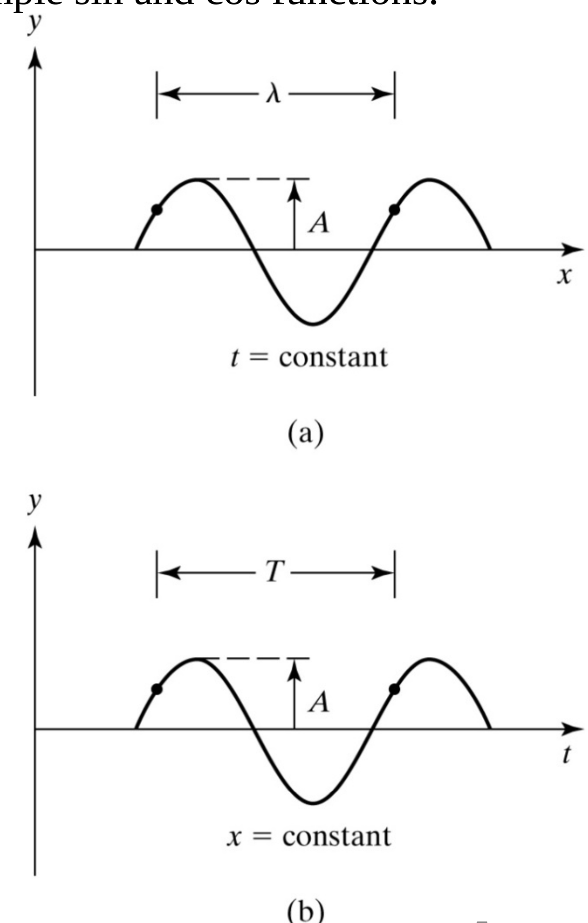
Period T: the repetitive temporal unit of the wave

$A\sin k\left[x + V\left(t+T\right)\right] = A\sin k\left(x+Vt\right) \to kVT = 2\pi$ or $VT/\lambda = 1$

Frequency ν: number of oscillations per unit time $\nu = 1/T \to \boxed{V = \nu\lambda}$

Angular frequency: $\omega = 2\pi\nu$ related to temporal period ν

Wave number: $\kappa = 1/\lambda$ number of waves per unit length



(a)

(b)

Phase, phase velocity and harmonic waves, initial phase

$$y = A\begin{matrix}\sin\\ \cos\end{matrix}\left[k\left(x \mp Vt\right)\right]$$

$$y = A\begin{matrix}\sin\\ \cos\end{matrix}\left[2\pi\left(\frac{x}{\lambda} \mp \frac{Vt}{\lambda}\right)\right] = A\begin{matrix}\sin\\ \cos\end{matrix}\left[2\pi\left(\frac{x}{\lambda} \mp \frac{t}{T}\right)\right]\left\{\begin{matrix} y = 0 \text{ for } x = 0 \text{ and } t = 0 \\ y = A \text{ for } x = 0 \text{ and } t = 0 \end{matrix}\right\}\text{Initial phase } = 0$$

$$y = A\begin{matrix}\sin\\ \cos\end{matrix}\left[\left(\frac{2\pi}{\lambda}x \mp 2\pi\nu t\right)\right] = A\begin{matrix}\sin\\ \cos\end{matrix}\left(\underbrace{kx \mp \omega t}_{Phase}\right)$$

Phase: argument of the sin or cosine function in a harmonic wave that is a function of space and time.

$$\phi = k\left(x \mp Vt\right) = 2\pi\left(\frac{x}{\lambda} \mp \frac{t}{T}\right) = kx \mp \omega t$$

For any set of x and t that ϕ stays constant the displacement $y = A\sin\left(\phi\right)$ is also constant.

Motion of a fixed point on a wavefront is described by motion of a constant phase point.

For a point with $\phi =$ constant we have

$$d\phi = 0 = kdx \mp Vdt \rightarrow \boxed{V_{phase} = \left.\frac{\partial x}{\partial t}\right|_{\phi}} \text{ General formula}$$

Phase velocity of a wave is speed of the motion of a constant-phase point on a disturbance.

Value of the pase velocity: for a harmonic wave at any given time $\phi = kx \mp \omega t + \varepsilon$

For constant phase: $\frac{\partial \phi}{\partial t} = k\left.\frac{\partial x}{\partial t}\right|_{\phi} \mp \omega = 0 \rightarrow V_{phase} = \left.\frac{\partial x}{\partial t}\right|_{\phi} \rightarrow \boxed{V_{phase} = \pm\frac{\omega}{k}}$ only for harmonic waves

Initial phase $= \phi_0 \rightarrow y = A\sin\left[k\left(x \mp Vt\right) + \phi_0\right]$; at $t = 0$ and $x = 0 \rightarrow y = A\sin\phi_0 = y_0 \rightarrow \phi_0 = \sin^{-1}\left(\frac{y_0}{A}\right)$

Exercise

For the wavefunction $\psi(x,t) = 10^3 \sin \pi(3\times10^6 x + 9\times10^8 t + 0.5)$

in SI units find the following quantities:

a) speed, b) wavelength, c) frequency, d) period, e) amplitude,

f) phase, g) initial phase, h) phase at $t = 10\ s$, i) phase velocity,

j) compare the speed and the phase velocity.

k) what is the direction of the motion of the wave. Does phse velocity indicate the direction of motion properly?

k) what is magnitude of the wave at $x = 0$ when $t = 0, \tau/4, \tau/2, 3\tau/4, \tau$?

l) plot the profile of the wave at $t = 0$ with initial pahse equal to $0,\ 0.5\pi,\ \pi,\ 2\pi.$

Complex numbers

More often wavefunctions are expressed in complex exponentials since complex exponentials can simplify the trigonometric expressions.

$z = z + jy$ or $z = x + iy$; $i \; or \; j = \sqrt{-1}$ and both x (real part) and y (imaginary part) are real numbers.

Absolute value or modulus: $|z| = \sqrt{x^2 + y^2}$;

Argument: $\tan\phi = \dfrac{y}{x}$

We can also write $z = Ae^{i\phi}$ and

$z^* = Ae^{-i\phi}$ the complex conjugate of z

Euler's formula: $e^{i\phi} = \cos\phi + i\sin\phi \rightarrow z = A(\cos\phi + i\sin\phi)$

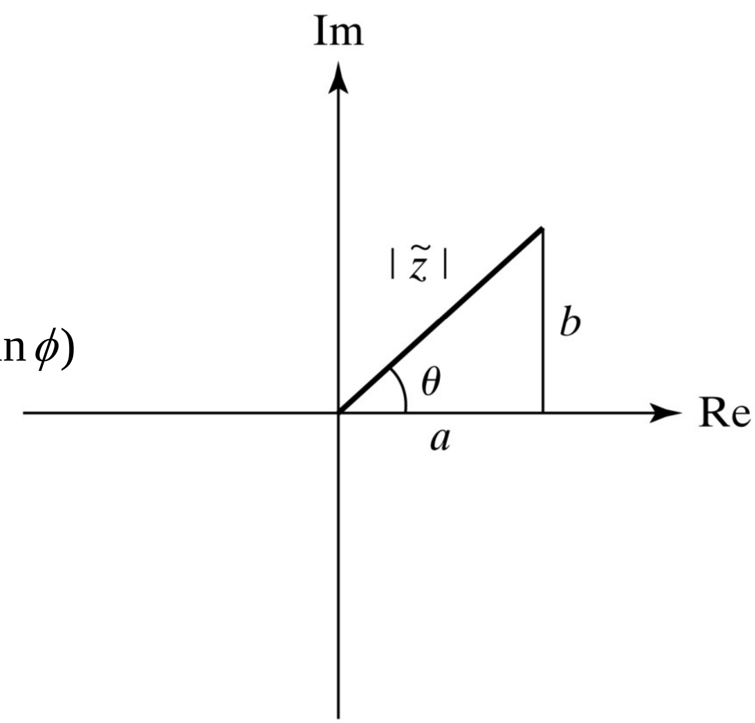
where $x = A\cos\phi$ and $y = A\sin\phi$

$A = |z| = \sqrt{x^2 + y^2}$ is the magnitude or amplitude

$\phi = \tan^{-1}\left(\dfrac{y}{x}\right)$ is the phase

Also $z^* = x - iy$ is the complex conjugate of $z = x + iy$.

$zz^* = \left(|z|e^{i\phi}\right)\left(|z|e^{-i\phi}\right) = |z|^2$



θ	$e^{i\theta}$
0	1
$\pi/2$	i
π	-1
$3\pi/2$	$-i$

Euler formula and harmonics waves as a complex function

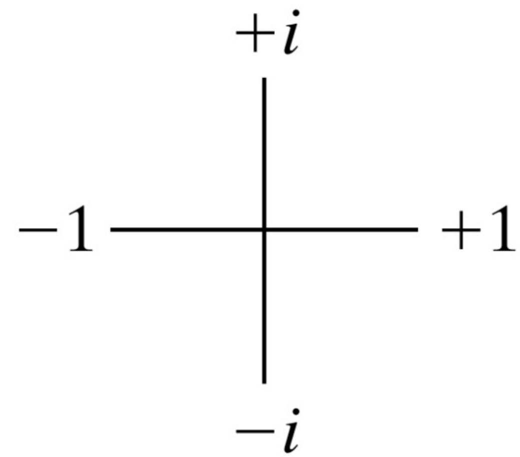
$$e^{i\phi} = \cos\phi + i\sin\phi$$

Using this formula we can express a harmonic wave as a real or imaginary part of a complex function.

$$\tilde{y} = Ae^{i(kx-\omega t)}$$

$$y = \mathrm{Re}\left(\tilde{y}\right) = A\cos\left(kx-\omega t\right)$$ or

$$y = \mathrm{Im}\left(\tilde{y}\right) = A\sin\left(kx-\omega t\right)$$



Any equatrion that involves linear terms in $\tilde{y}$ and its derivatives will hold for $y = \mathrm{Re}\left(\tilde{y}\right)$ and $y = \mathrm{Im}\left(\tilde{y}\right)$

Plane waves

Wavefront: a surface over which the phase of a wave is constant.

Plane waves have planar wavefronts that are surfaces perpendicular to the direction of propagation k. We write equation of a plane that is passing through $p_0(x_0,\ y_0,\ z_0)$ and is perpendicular to $\mathbf{k}$ (direction of propagation).

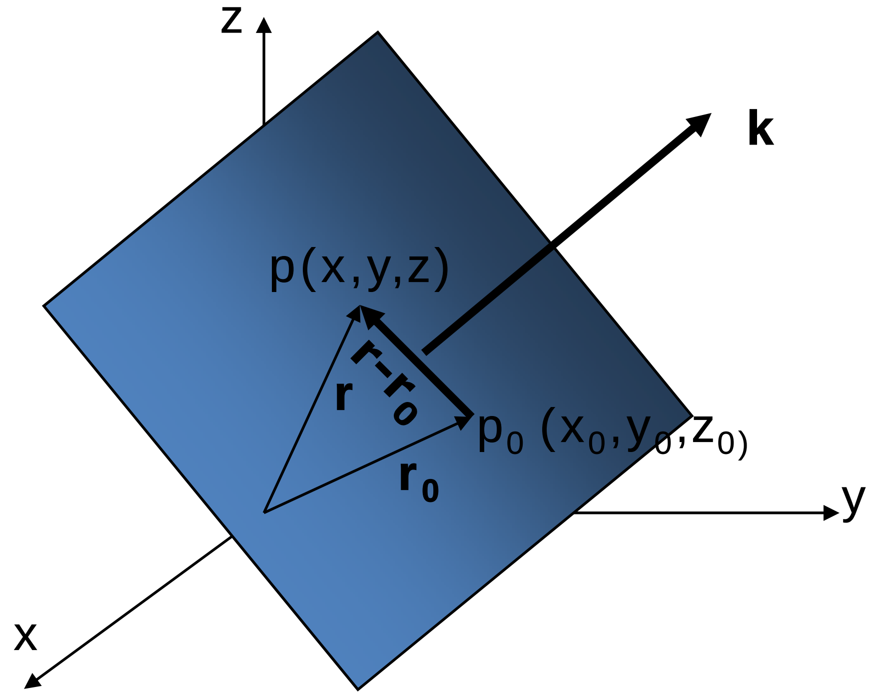
Consider an arbitrary point p on the planar wavefront, we must have:

$$(\mathbf{r}-\mathbf{r_0})\bullet\mathbf{k}=\mathbf{0} \;\rightarrow\; \mathbf{k}\bullet\mathbf{r}=\mathbf{k}\bullet\mathbf{r_0}=constnt$$

is equation of such a plane perpendicular to $\mathbf{k}$

$\psi(\mathbf{r})=A\sin(\mathbf{k}\bullet\mathbf{r})$ is a function defined on a family of planes all perpendicular to $\mathbf{k}$.

Phase and $\psi(\mathbf{r})$ are constant on one plane (wavefront) but they vary sinusoidally from plane to plane.



The progressive wave equation is then:

$\psi(\mathbf{r},t)=A\sin(\mathbf{k}\bullet\mathbf{r}\mp\omega t)$ or

$$\psi(\mathbf{r},t)=Ae^{i(\mathbf{k}.\mathbf{r}\mp\omega t)}=A\cos(\mathbf{k}\bullet\mathbf{r}\mp\omega t)+iA\sin(\mathbf{k}\bullet\mathbf{r}\mp\omega t)$$

$\psi(\mathbf{r},t)=Ae^{i(kr\cos\theta\mp\omega t)}=Ae^{i(ks\mp\omega t)}$ where s is the component of r along the direction of propagation or k. Amplitude of a plane wave stays constant as it propagates.

Harmonic waves in 3-dimension

A general harmonic wave in three-dimension in complex form:

$$\psi(\mathbf{r},t) = Ae^{i(\mathbf{k}\bullet\mathbf{r}-\omega t)}$$

The 3-D wave equation that satisfies $\psi(\mathbf{r},t)$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{V^2}\frac{\partial^2\psi}{\partial t^2}$$

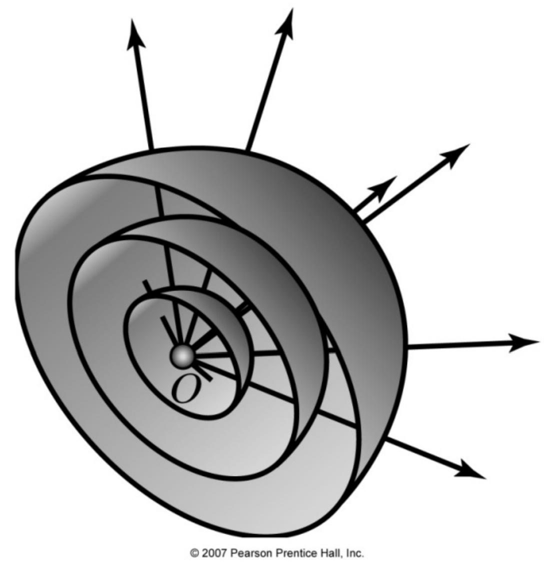
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi=\frac{1}{V^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$\boxed{\nabla^2\psi=\frac{1}{V^2}\frac{\partial^2\psi}{\partial t^2}}$$ 3-D wave equation

Spherical waves

By solving the differential spherical wave equation we can arrive at harmonic spherical wave function:



$$\psi(r,t)=\frac{A}{r}\sin k(r\mp Vt)$$

$$\psi(r,t)=\frac{A}{r}e^{ik(r\mp Vt)}=\frac{A}{r}e^{i(kr\mp\omega t)}$$

that represents cluster of concentric shperes at any instance.

On each sphere r is constant so $\psi(r,t)$ is constant.

Here A, the source strength is a constant.

$\frac{A}{r}$ is the amplitude and it varies inversly with the distance from the source to conserve the energy. The irradiance in $\frac{W}{m^2}\propto\left(\frac{A}{r}\right)^2$ is the iverse square law of propagation for the spherical waves.

Same as plane waves, for spherical waves

$$k=\frac{2\pi}{\lambda},\quad \omega=\frac{2\pi}{T}$$

Cylindrical Waves

$$\psi = \frac{A}{\sqrt{p}} e^{i(k\rho \mp \omega t)}$$

ρ is the perpendicular distance from the line of symmetry of the cylindrical wave.

If the the line of sylmmetry is the z axis then

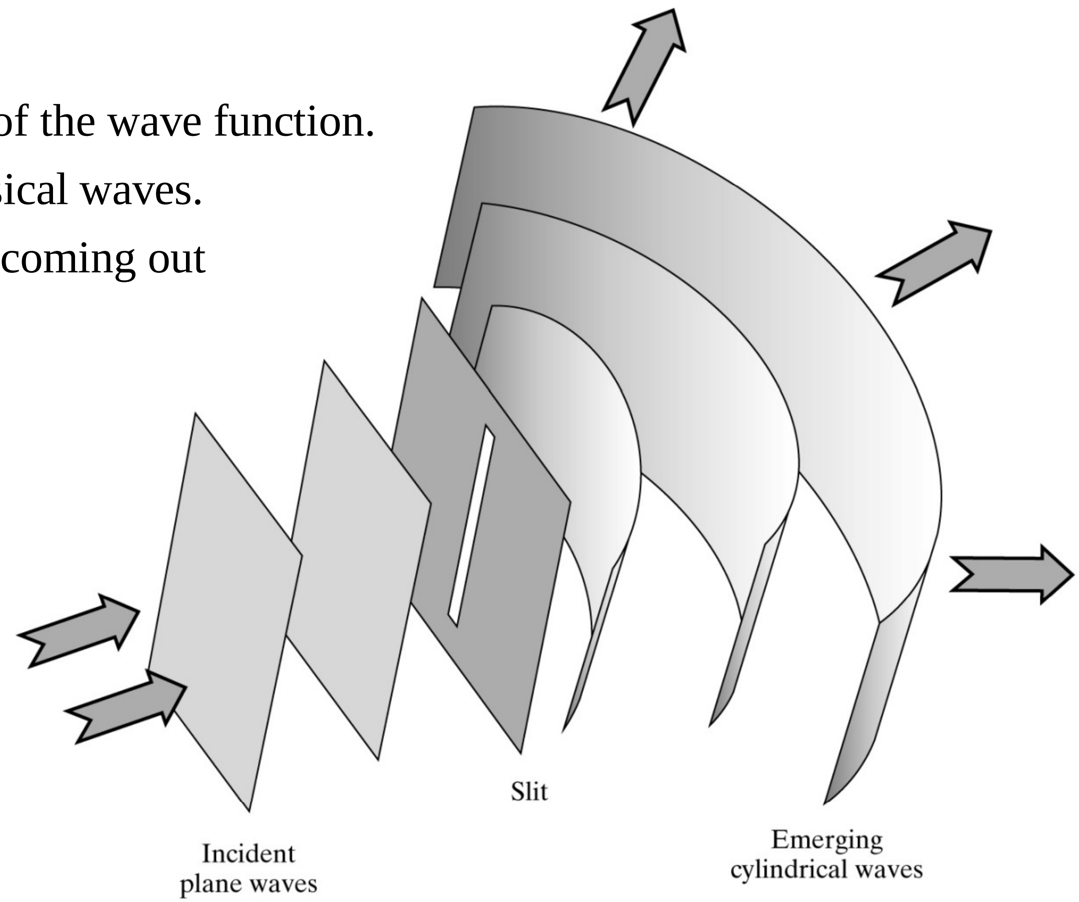
$$\rho = \sqrt{x^2 + y^2}$$

These waves are not exact solutions of the wave function.

So they do not exactly represent physical waves.

But they are useful to express waves coming out

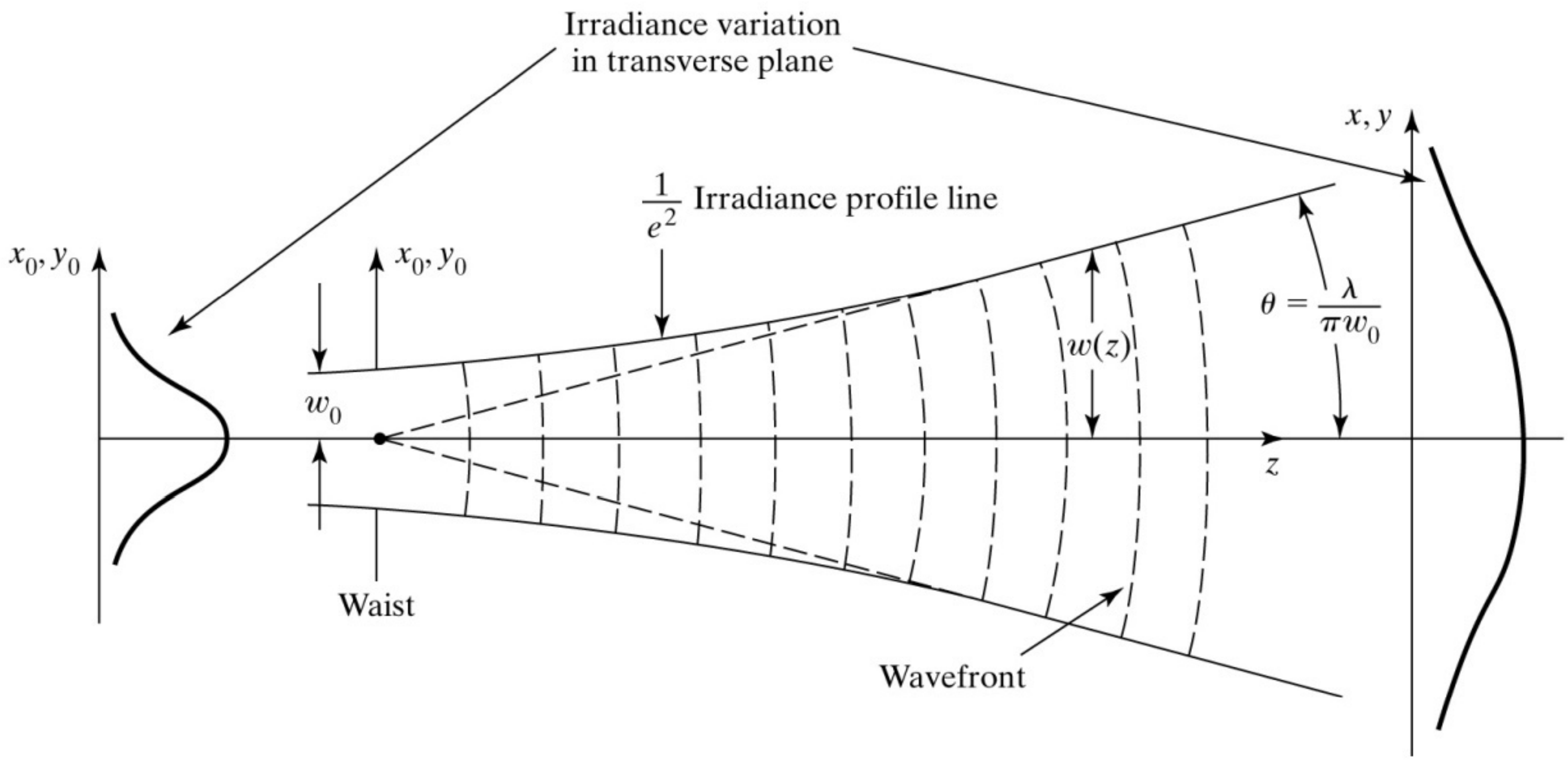
of a slit illuminated by a plane wave.



Hermite-Gaussian waves

A good representation of the laser beams and an approximate solution to the wave equation.

Good for the cases that the irradiance is strongly confined to the direction of propagation.

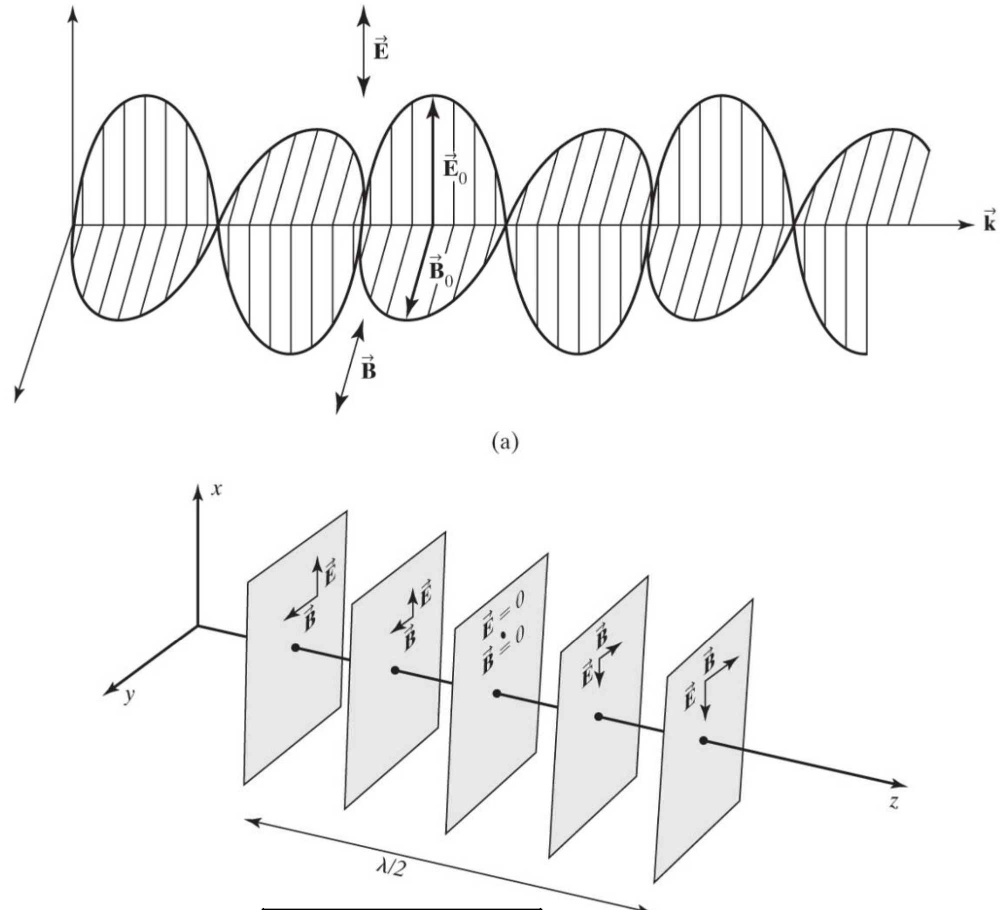


Electromagnetic waves

- The harmonic wave equation can represent any type of disturbance with sinusoidal behavior.
- Physical significance of the disturbance is different for different systems (pressure, displacement ...)
- Maxwell showed that light is composed of electric and magnetic fields oscillating perpendicular to each other and propagating in the direction k, perpendicular to the plane of oscillations.
- For light waves the disturbance is the magnitude of varying electric or magnetic fields that are described with the following harmonic wavefunctions:

$$\mathbf{E} = \mathbf{E_0} e^{i(\mathbf{k}.\mathbf{r}-\omega t)} = \mathbf{E_0}\sin(\mathbf{k}.\mathbf{r}-\omega t),$$

$$\mathbf{B} = \mathbf{B_0} e^{i(\mathbf{k}.\mathbf{r}-\omega t)} = \mathbf{B_0}\sin(\mathbf{k}.\mathbf{r}-\omega t)$$



Electric fields are generated by electric charges and time-verying magnetic fields.
Magnetic fields are generated by electric currents (charge in motion) and time-varying electric fields.
This intedependence of the $\mathbf{E}$ and $\mathbf{B}$ is a key point in description of the light.
Lorentz force: an electric charge q, moving with velocity of $\mathbf{v}$ in an area that contains both $\mathbf{E}$ and $\mathbf{B}$ fileds, feels forces due to existance of both fields. $\boxed{\mathbf{F} = q\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B})}$

Before Maxwell

Faraday's induction law: a time-varying magnetic field will have an electric field associated with it. $\oint_C \mathbf{E}\cdot d\vec{l} = -\iint_{\mathbf{A}} \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}$

Gauss's law-electric: $\oiint_{\mathrm{A}} \mathbf{E}\cdot d\mathbf{s} = \frac{1}{\varepsilon_0}\iiint_V \rho dV,$ when there are no sources or sinks of the electric field within the region encompassed by a closed surface, the net flux through the surface equals to zero.

Gauss's law-magnetic: $\Phi_{\mathrm{M}} = \oiint_{\mathrm{A}} \mathbf{B}\cdot d\mathbf{s} = 0,$ there is no magnetic monopole

Ampere's circuital law: $\oint_C \mathbf{B}\cdot d\vec{l} = \mu\iint_{\mathbf{A}}\left(\mathbf{J} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}\right)\cdot d\mathbf{s},$ a time-varying $\mathbf{E}$-field or charges in motion (electric current) will generate a $\mathbf{B}$-field

Maxwell equations; integral form

Behavior of electric and magnetic fields in a medium with electric permitivity $\varepsilon,$ and magnetic permeability μ, in presence of free charges ρ, and current density $\mathbf{J},$ is explained by four integral equations known as Maxwell equations.

$$\oint_C \mathbf{E}\cdot d\vec{l} = -\iint_{\mathbf{A}} \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} \qquad \leftarrow \text{Farady}$$

$$\oint_C \frac{\mathbf{B}}{\mu}\cdot d\vec{l} = \iint_{\mathbf{A}} \left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right)\cdot d\mathbf{s} \qquad \leftarrow \text{Amper}$$

$$\oiint_{\mathrm{A}} \mathbf{B}\cdot d\mathbf{s} = 0 \qquad \leftarrow \text{Gauss magnetic}$$

$$\oiint_{\mathrm{A}} \varepsilon\mathbf{E}\cdot d\mathbf{s} = \iiint_V \rho dV \qquad \leftarrow \text{Gauss electric}$$

Maxwell equations; differential form

$$\oint_C \mathbf{E} \cdot d\vec{l} = - \iint_{\mathbf{A}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \qquad \to \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_C \frac{\mathbf{B}}{\mu} \cdot d\vec{l} = \iint_{\mathbf{A}} \left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{s} \qquad \to \nabla \times \mathbf{B} = \mu \left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\oiint_{\mathrm{A}} \mathbf{B} \cdot d\mathbf{s} = 0 \qquad \to \nabla \cdot \mathbf{B} = 0$$

$$\oiint_{\mathrm{A}} \varepsilon \mathbf{E} \cdot d\mathbf{s} = \iiint_V \rho dV \qquad \to \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$$

We used the following theorems: Stoke's $\oint_C \mathbf{E} \cdot d\vec{l} = \iint_{\mathbf{A}} \nabla \times \mathbf{E} \cdot d\mathbf{s}$

Gauss's $\oiint_{\mathrm{A}} \mathbf{B} \cdot d\mathbf{s} = \iiint_V \nabla \cdot \mathbf{B} dV$

Constitutive relations

$\mathbf{H} =$ Magnetic field; $\mathbf{B} =$ Magnetic induction (effect of the $\mathbf{H}$ in the medium $\mathbf{B} = \mu\mathbf{H}$);

$\mathbf{E} =$ Electric field; $\mathbf{D} =$ Electric displacement (effect of the $\mathbf{E}$ in the medium $\mathbf{D}=\varepsilon\mathbf{E}$);

$\mathbf{J} =$ Current density;

Constituitive relations are: $\mathbf{D} = \mathbf{D}(\mathbf{E},\mathbf{B})$; $\mathbf{H} = \mathbf{H}(\mathbf{E},\mathbf{B})$; $\mathbf{J} = \mathbf{J}(\mathbf{E},\mathbf{B})$

These relations may be nonliner or depend on the past (hysteresis).

Linear response: the applied fields are small so they induce electric and magnetic polarizations proportional to the magnitude of the applied field

(ferroelectric and ferromagnetic material are exceptions are nonlinear material)

$\mathbf{D}_\alpha = \sum_\beta \varepsilon_{\alpha\beta} E_\beta$; $\quad \mathbf{H}_\alpha = \sum_\beta \mu^{-1}_{\alpha\beta} B_{\alpha\beta}$ α, β are the coordinates x,y,z.

$\varepsilon_{\alpha\beta}$ is electric permitivity or dielectric tensor;

$\mu^{-1}_{\alpha\beta}$ is inverse magnetic permitivity tensor

For material isotropic in space both ε and μ are diagonal matrices and all elements are equal. For isotropic material $\mathbf{D}=\varepsilon\mathbf{E}$ and $\mathbf{H}=\mu^{-1}\mathbf{B}$

At high enough fields every material is nonlinear (nonlinear optics)

$$D_\alpha = \sum_\beta \varepsilon^{(1)}_{\alpha\beta} E_\beta + \sum_{\beta,\gamma} \varepsilon^{(2)}_{\alpha\beta\gamma} E_\beta E_\gamma + ...$$

Electric field in medium

1) $\mathbf{P}$ polarization vector: $\underline{\mathbf{P} = \varepsilon_0 \chi \mathbf{E}}$ electric dipole moment per unit volume.

2) $\mathbf{D}$ displacement field: $\underline{\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}}$ electric field within the material

3) $\mathbf{E}$ internal electric field: $\mathbf{E} = \dfrac{\mathbf{D}}{\varepsilon_{\mathbf{0}}} - \dfrac{\mathbf{P}}{\varepsilon_0}$ and we have $\underline{\mathbf{D} = \varepsilon(\mathbf{E})\ \mathbf{E}}$

$\mathbf{D}$ and $\mathbf{E}$ lines begin and end on free charges or polarization cahrges.

In absence of free charge field lines close on temselves

$(\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{D} = 0)$.

For homogeneous, linear, isotropic dielectrics $\mathbf{P}$ and $\mathbf{E}$ are in the same direction so $\mathbf{D} = \varepsilon \mathbf{E}$, where $\varepsilon = \varepsilon_0 (1 + \chi)$, and $\mathrm{K}_e = \varepsilon_r = \varepsilon / \varepsilon_0 = 1 + \chi$ is the relative dielectric constant or function.

4) $\mathbf{J}$ current density: acording to the Ohm's law $\left(I = V / R \right)$ electric field intensity determines the flow of the cahrge in a conductor $\underline{\mathbf{J} = \sigma \mathbf{E}}$, true for conductors at constant temperature.

Constitutive relations: magnetic fields in a medium

1) $\mathbf{M}$ magnetic polarization vector: $\mathbf{M} = \mathrm{K}_m \mathbf{H}$.

2) $\mathbf{H}$ magnetic field intensity: $\underline{\mathbf{H} = \mu_{\mathbf{0}}^{-1}\mathbf{B} - \mathbf{M}}$,

<u>For homogeneous, linear (nonferromagnetic), isotropic</u> medium $\mathbf{B}$ and $\mathbf{H}$ are parallel and proportional,

$\underline{\mathbf{H} = \mu^{-1}\mathbf{B}}$ and $\mu = \mu_0(1+\mathrm{K}_m)$ and $\mu_r = \dfrac{\mu}{\mu_0} = (1+\mathrm{K}_m)$

For most optical material $\mu_r = 1$ or $\mathrm{K}_m = 0$ or no magnetization occures under magnetic field.

Maxwell equations using the constitutive relations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \rightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \rightarrow \nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{B} = \mu\left(\mathbf{J} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}\right) \rightarrow \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \nabla \cdot \mathbf{B} = 0$$

The wave equation for the E and M components of the EM waves

Maxwell equations in nonconducting medium ($\rho = 0,\ \mathbf{J} = 0$) vacuum

$$\varepsilon = \varepsilon_0,\ \mu = \mu_0, \qquad (1)\ \ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad (2)\ \ \nabla\times\mathbf{B} = \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$(3)\ \ \nabla\cdot\mathbf{B} = 0, \qquad (4)\ \ \nabla\cdot\mathbf{E} = 0$$

Take curl of (1) and use (2) to eliminate $\mathbf{B}$, $\nabla\times\nabla\times\mathbf{E} = -\dfrac{\partial}{\partial t}(\nabla\times\mathbf{B})$,

$\nabla\times\nabla\times\mathbf{E} = -\mu_0\varepsilon_0\dfrac{\partial^2\mathbf{E}}{\partial t^2},\ \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\mu_0\varepsilon_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}$, using (4) we get

differential wave equation: $\boxed{\nabla^2\mathbf{E} = \mu_0\varepsilon_0\dfrac{\partial^2\mathbf{E}}{\partial t^2} = \dfrac{1}{\mathbf{V}^2}\dfrac{\partial^2\mathbf{E}}{\partial t^2},}$ with $\boxed{|\mathbf{V}| = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}}$

We can also show: $\boxed{\nabla^2\mathbf{B} = \mu_0\varepsilon_0\dfrac{\partial^2\mathbf{B}}{\partial t^2} = \dfrac{1}{\mathbf{V}^2}\dfrac{\partial^2\mathbf{B}}{\partial t^2},}$

The index of refraction

Velocity of light based on Maxwell's theoretical treatment in vacuum is

$c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$ and in medium is $V = \dfrac{1}{\sqrt{\varepsilon \mu}}$.

Absolute index of refraction defined as: $n = \dfrac{c}{V} = \sqrt{\dfrac{\varepsilon \mu}{\varepsilon_0 \mu_0}} = \sqrt{K_e \mu_r} = \sqrt{\varepsilon_{\mathrm{r}} \mu_r}$.

The $\mathbf{E}$ field polarizes the medium rusulting in change of the displacement field.

$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$.

The result is a change in ε and consequently a change in n and V, speed of light in the medium. $\varepsilon(\omega)$ and $n(\omega)$ are functions of the frequency of the EM waves.

Usually $\mu \approx \mu_0$ so $n = \sqrt{\dfrac{\varepsilon}{\varepsilon_0}} = \sqrt{K_e} = \sqrt{\varepsilon_{\mathrm{r}}}$; K_e is the complex dielectric function.

Physical meaning of the index of refraction I

Consider a general case where n is a complex number $n = n' + in''$. Consider the E component of a plane wave,

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$\mathbf{k}$, the propagation vector in medium is complex.

$\mathbf{k}_0$ is propagation vector in vacuum.

With $V_P = \dfrac{\omega}{|\mathbf{k}|}$, the phase velocity, and $c = \dfrac{\omega}{|\mathbf{k}_0|} = nV$

$\rightarrow |\mathbf{k}| = n|\mathbf{k}_0| = \dfrac{\omega}{V} = \dfrac{\omega n}{c}$ for one-dimensional case

$$E = E_{0y} e^{i(kx - \omega t)} = E_{0y} e^{i(\frac{n\omega}{c}x - \omega t)} = E_{0y} e^{i(\frac{(n'+in'')\omega}{c}x - \omega t)} = E_{0y} e^{-\frac{n''\omega x}{c}} e^{i\omega(\frac{n'x}{c} - t)}$$

Physical meaning of the index of refraction II

$$E = E_{0y} e^{-\frac{n''\omega x}{c}} e^{i\omega(\frac{n'x}{c} - t)}$$

$e^{-\frac{n''\omega x}{c}}$ is a real term and decays exponentially as wave propagates.

$e^{i\omega(\frac{n'x}{c} - t)}$ has a harmonic wave form and propagates without loss

This suggests that

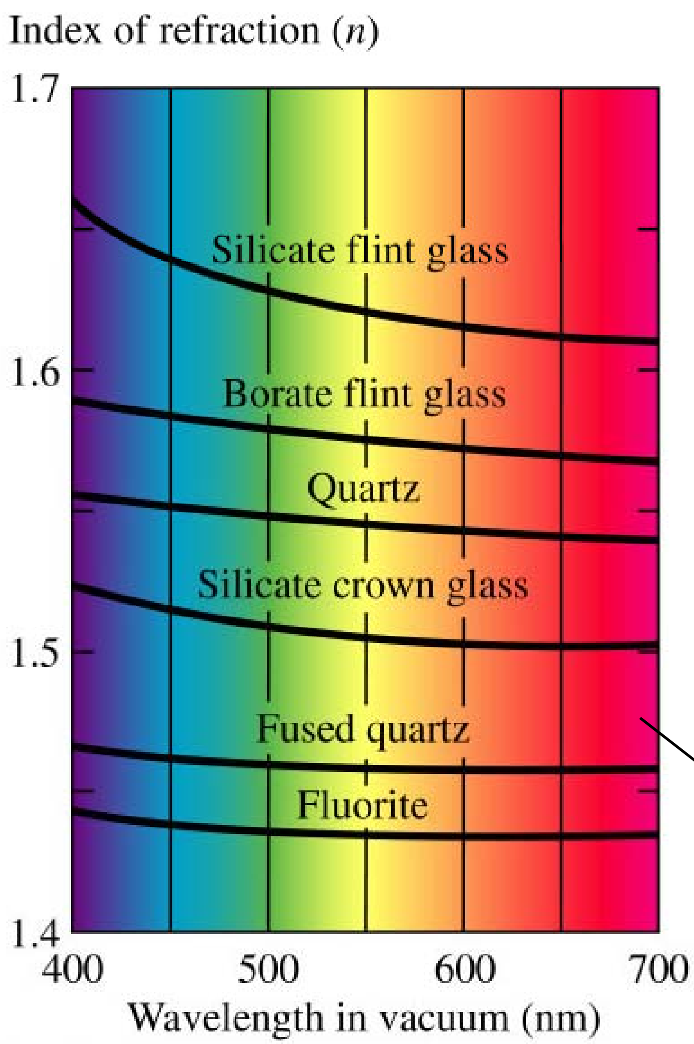
n'', the imaginary part of n is associated with absorption.

n', the real part of n is associated with propagation.

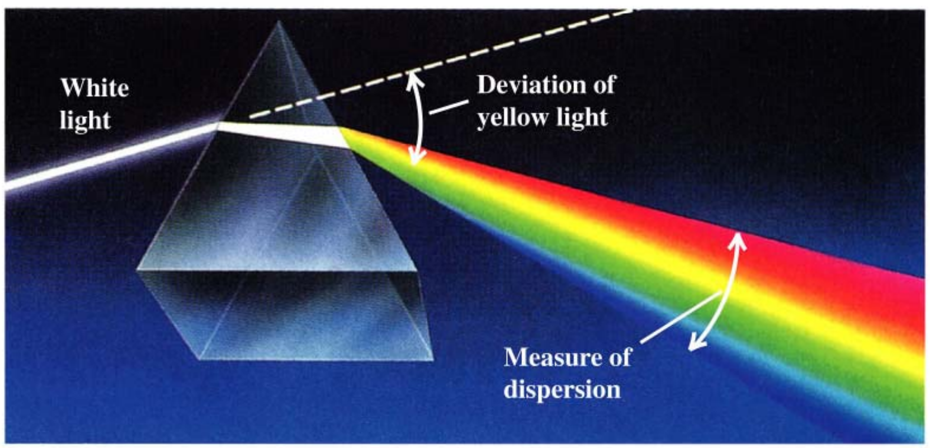
In absence of n'', the case in many dielectrics in the visible part of the EM spectrum, $n = n' = c/V$ is simply the ratio of the speeds in vacuum and in the medium.

$n = n' = c/V$ is also called absolute index of refraction.

Frequency dependence of the index of refraction

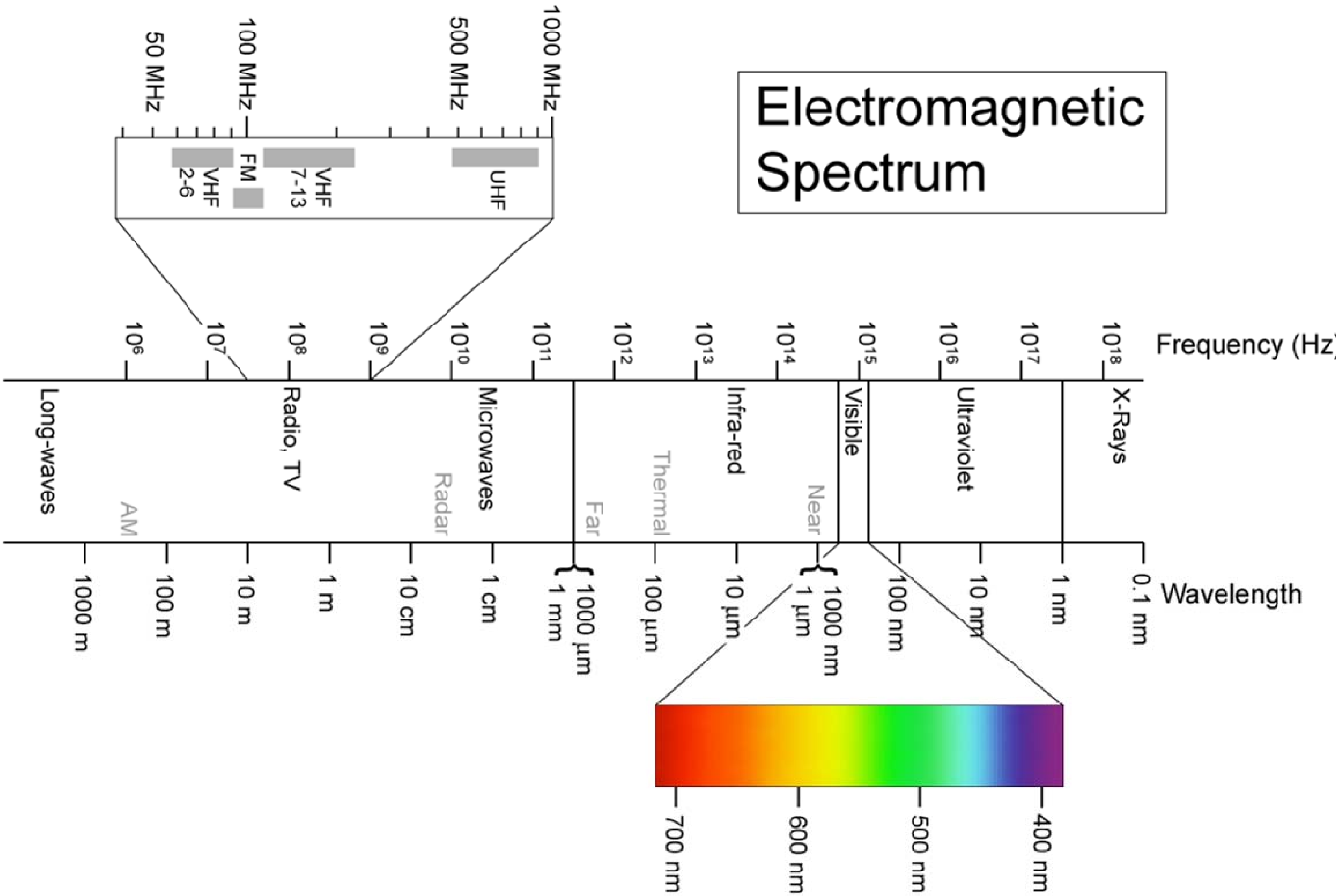


publishing as Addison Wesley.



The wavelength dependence of n is stronger at short wavelengths or high frequencies.
For most dielectrics the imaginary part of the n is negligible in the visible band

The electromagnetic spectrum



Energy and momentum I

Physical manifestation of electromagnetic waves is their enegy and momentum.

Energy density u, is radiant energy per unit volume

In order to calculate the energy content of the EM waves we start from

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=\mathbf{H}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{H})$$

Using $\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}$ and $\nabla\times\mathbf{H}=\mathbf{J}+\dfrac{\partial\mathbf{D}}{\partial t}$ and $\mathbf{J}=\sigma\mathbf{E}$

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=\mathbf{H}\cdot(-\frac{\partial\mathbf{B}}{\partial t})-\mathbf{E}\cdot(\frac{\partial\mathbf{D}}{\partial t})-\mathbf{E.J}=\mathbf{H}\cdot(-\frac{\partial\mathbf{B}}{\partial t})-\mathbf{E}\cdot(\frac{\partial\mathbf{D}}{\partial t})-\sigma\mathbf{E.E}$$

$$\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}=\mathbf{H}\cdot\frac{\partial(\mu\mathbf{H})}{\partial t}=\frac{1}{2}\frac{\partial(\mu\mathbf{H.H})}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\mathbf{E}\cdot\frac{\partial(\varepsilon\mathbf{E})}{\partial t}=\frac{1}{2}\frac{\partial(\varepsilon\mathbf{E.E})}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)$$

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=-\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right)-\sigma E^2$$

Energy and momentum II

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2$$

Integrating both sides over a <u>volume V</u>, and applying the divergence theorem $\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dV = \oint_S \mathbf{E} \times \mathbf{H} \cdot d s$

$$\underbrace{\oint_S \mathbf{E} \times \mathbf{H} \cdot d\mathbf{s}}_{\substack{\text{power rate leaving the} \\ \text{volume from its surface}}} = -\underbrace{\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) dV}_{\substack{\text{time-rate of change of energy stored} \\ \text{in electric and magnetic fields}}} - \underbrace{\int_V \sigma E^2 dV}_{\substack{\text{ohmic power dissipated in the} \\ \text{volume as a result of} \\ \text{conduction current density } \sigma\mathbf{E} \\ \text{in presence of } \mathbf{E}}}$$

Thus the $\mathbf{E} \times \mathbf{H}$ is a vector representing <u>power flow per unit volume</u>.

Poynting's Theorem

$\mathbf{S} = \mathbf{P} = \mathbf{E} \times \mathbf{H} = \frac{1}{\mu}\ \mathbf{E} \times \mathbf{B} \quad (W/m^2)$ is known as Poynting vector.

$|\mathbf{S}| =$ power density crossing a surface whose normal is parallel to $\mathbf{S}$.

Poynting's theorem: surface integral of $\mathbf{P}$ or $\mathbf{S}$ over a closed surface $S,$ equals the power leaving the enclosed volume V by S.

$$-\oint_S \mathbf{S}.d\mathbf{s} = \frac{\delta}{\delta t}\int_V (u_e + u_m)dV + \int_V P_\sigma dV \quad \text{where}$$

$u_e = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon \mathbf{E}\cdot\mathbf{E}^* = \frac{1}{2\varepsilon}\mathbf{D}\cdot\mathbf{D}^* =$ Electric energy density

$u_e = \frac{1}{2}\mu H^2 = \frac{1}{2}\mu \mathbf{H}\cdot\mathbf{H}^* = \frac{1}{2\mu}\mathbf{B}\cdot\mathbf{B}^* =$ Magnetic energy density

$P_\sigma = \sigma E^2 = \sigma \mathbf{E}\cdot\mathbf{E}^* =$ Ohmic power density

Exercise

RV1-16) a) Prove that for a harmonic plane electromagnetic wave with the following $\mathbf{E}$ field component traveling through an insulating isotropic medium, $|\mathbf{E}| = V|\mathbf{B}|$. In vacuum $|\mathbf{E}| = c|\mathbf{B}|$.

$\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$

b) Calculate the magnitude of the poynting vector

c) Prove that the energy content of this plane wave, in unit volume, due to electric and magnetic fields is equal.

d) Calculate the total energy per unit volume.

Irradiance

Irradiance I, or the amount of light: average energy over unit area per unit time. I is independent of detector area A, and duration of measurement.

Time averaged value of the Poynting vector $\mathbf{S}$, or irradiance is:

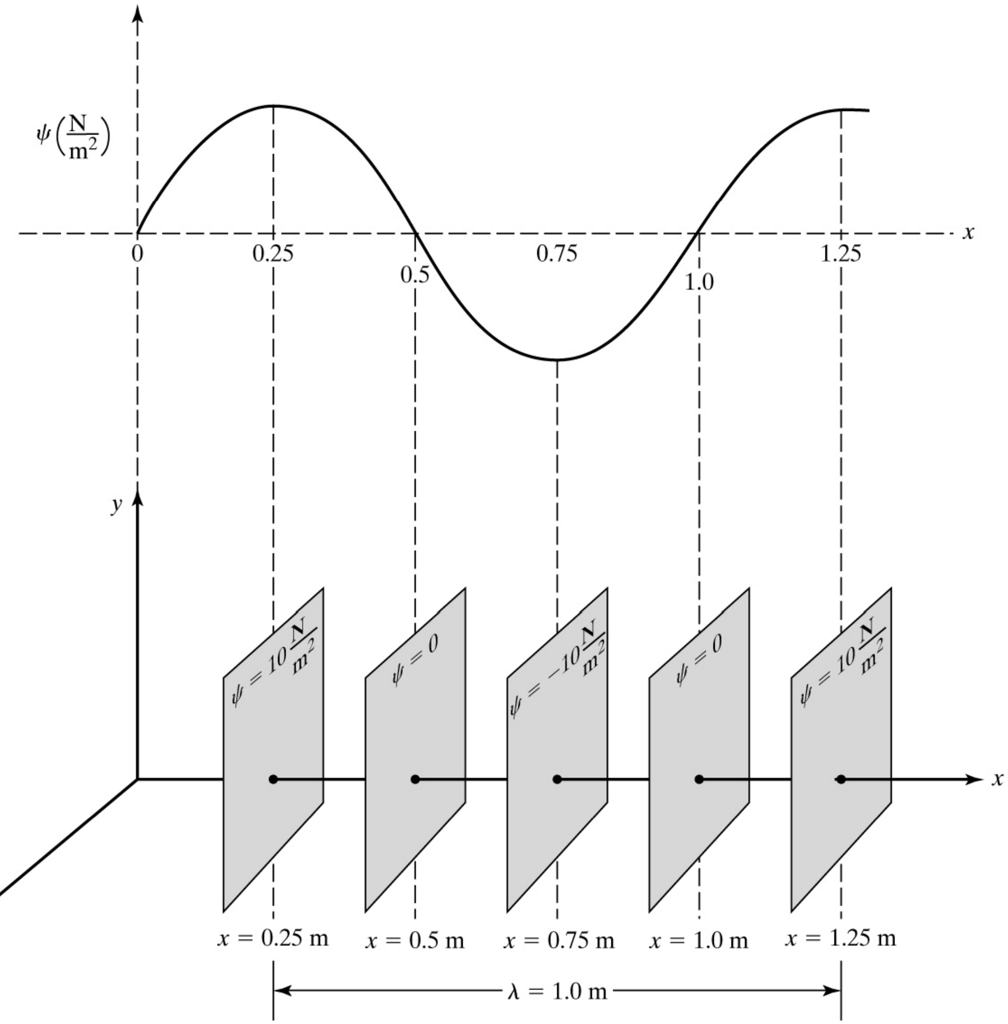
$$I \equiv \left\langle \mathbf{S} \right\rangle_T = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} (\mathbf{E} \times \mathbf{H}) dt$$

Note that the averaging time $T \gg \tau$, has to be much greater than the period.

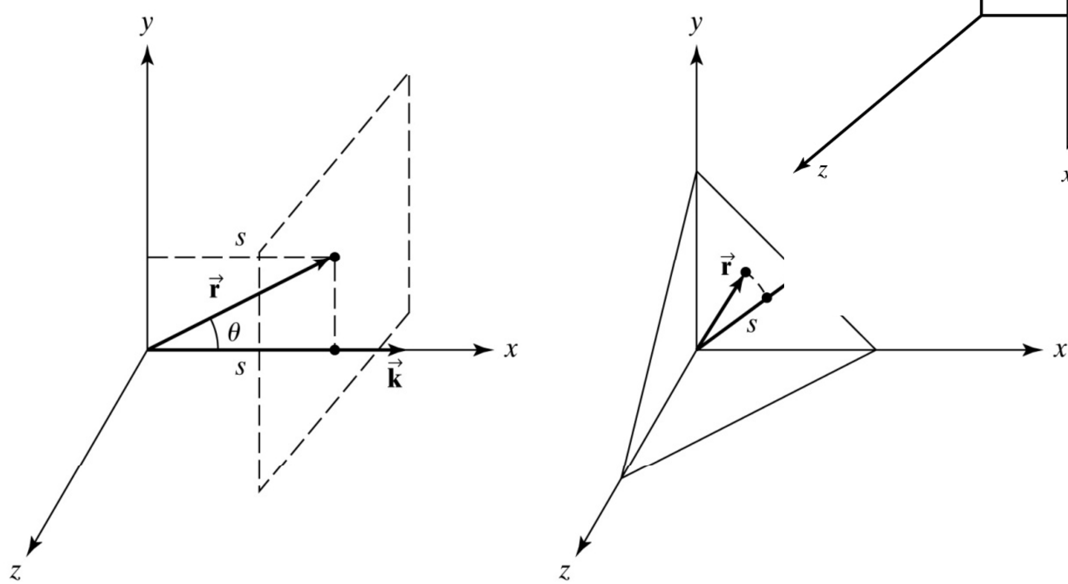
Irradince is proportional to the square of the amplitude of the $\mathbf{E}$ field.

For linear, homogeneous, isotropic dielectric

$I = \varepsilon V \left\langle E^2 \right\rangle_T$ In vacuum: $I = \frac{c}{\mu_0} \left\langle B^2 \right\rangle_T = \varepsilon_0 c \left\langle E^2 \right\rangle_T$

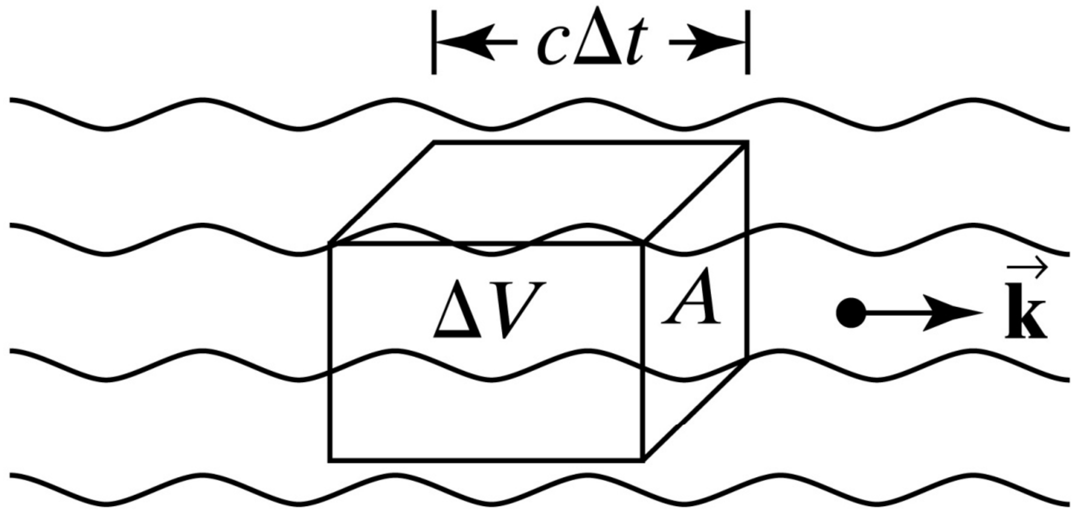


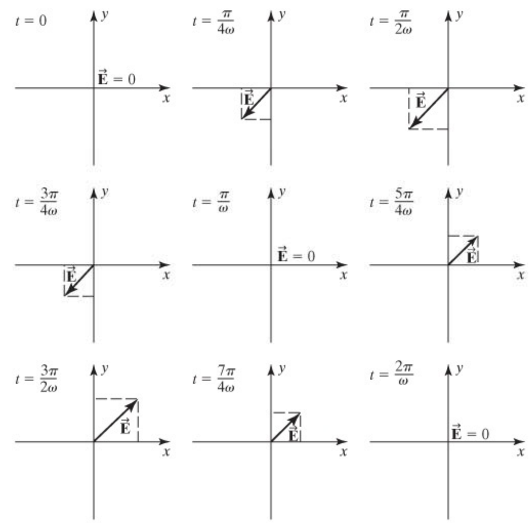
ψ (N/m²)
x
0
0.25
0.5
0.75
1.0
1.25
y
x
z
ψ = 10 N/m²
ψ = 0
ψ = −10 N/m²
ψ = 0
ψ = 10 N/m²
x = 0.25 m
x = 0.5 m
x = 0.75 m
x = 1.0 m
x = 1.25 m
λ = 1.0 m



y
s
r
θ
s
k
x
z
y
r
s
x
z

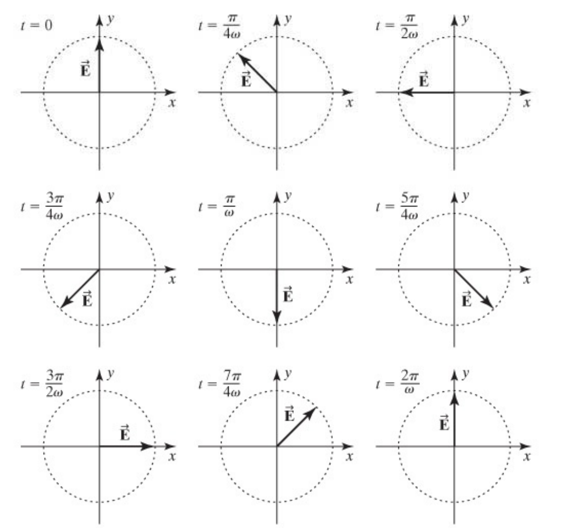
(a) (b)


cΔt
ΔV
A
$\vec{\mathbf{k}}$



$\vec{\mathbf{E}} = E_0 \sin(kz - \omega t)\hat{\boldsymbol{x}} + E_0 \sin(kz - \omega t)\hat{\boldsymbol{y}}$

(a)



$\vec{\mathbf{E}} = E_0 \sin(kz - \omega t)\hat{\boldsymbol{x}} + E_0 \cos(kz - \omega t)\hat{\boldsymbol{y}} = E_0 \sin(kz - \omega t)\hat{\boldsymbol{x}} + E_0 \sin(kz - \omega t + \pi/2)\hat{\boldsymbol{y}}$

(b)