# Chapter 18
# Matrix Methods in Paraxial Optics

Lecture Notes for Modern Optics based on
Pedrotti & Pedrotti & Pedrotti
Instructor: Nayer Eradat
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# Matrix methods in paraxial optics

- Describing a single thick lens in terms of its cardinal points.
- Describing a single optical element with a 2x2 matrix.
- Analysis of train of optical elements by multiplication of 2x2 matrices describing each element.
- Computer ray-tracing methods, a more systematic approach

## Cardinal points and cardinal planes

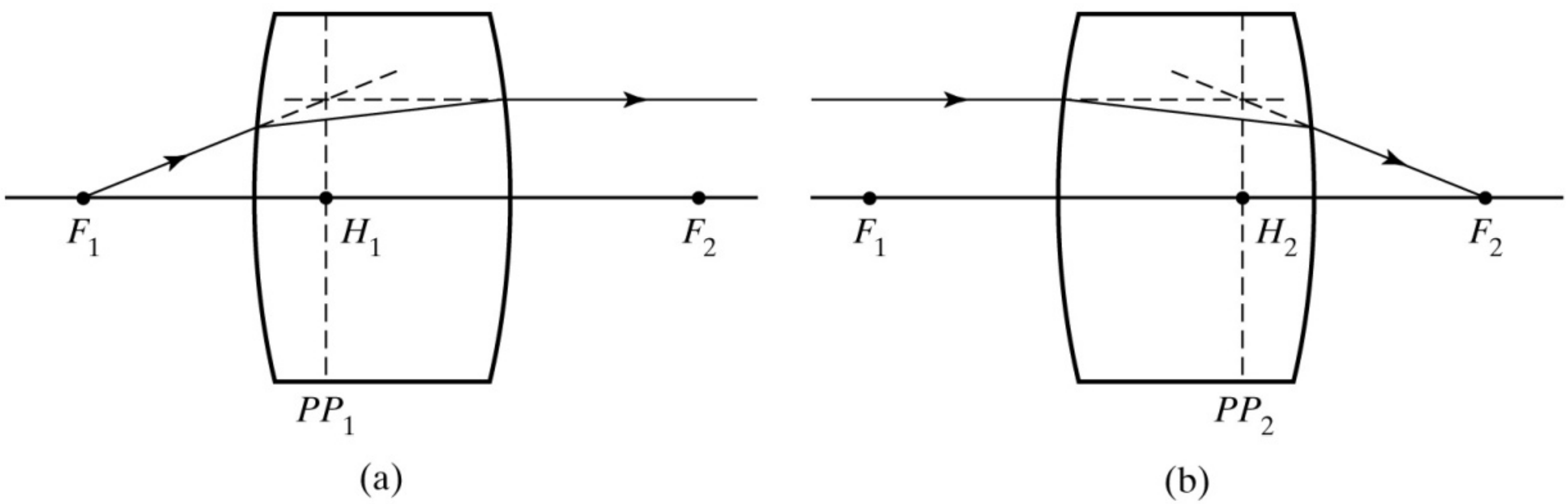
We define six **cardinal points** on the axis of a thick lens from which its imaging properties can be deduced.

Planes normal to the axis at the cardinal points are called **cardinal planes**.

Cardinal points and planes include

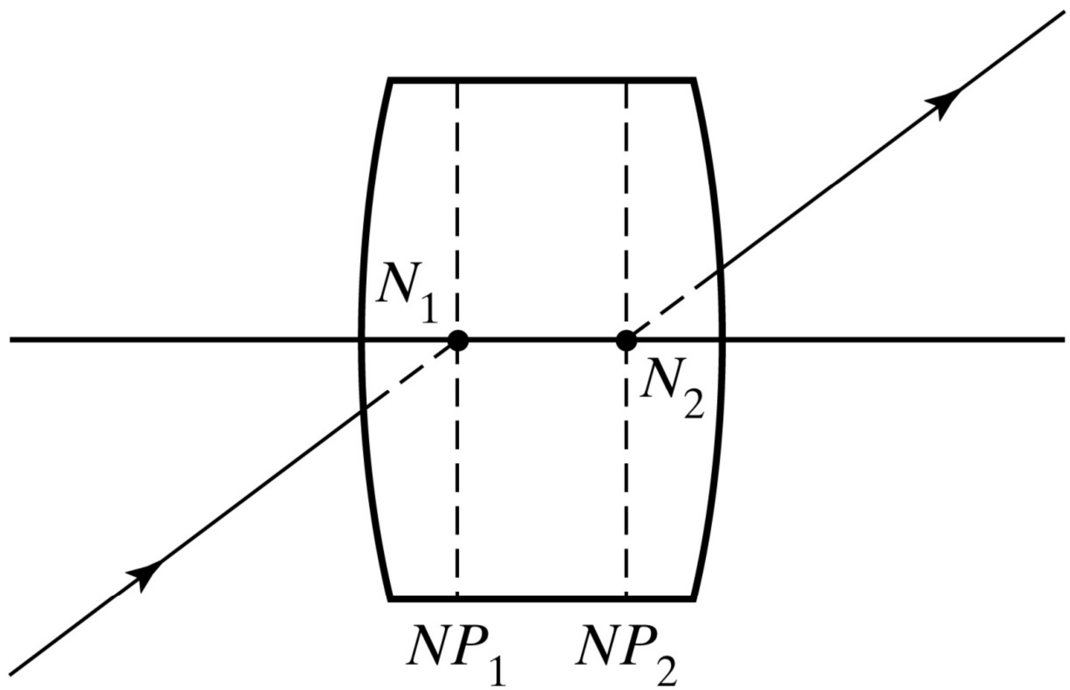
**First and second set** of **focal points** and **focal planes.**

**First and second principal points** and **principal planes.** The rays determining the focal points change direction at their intersection with the principal planes.



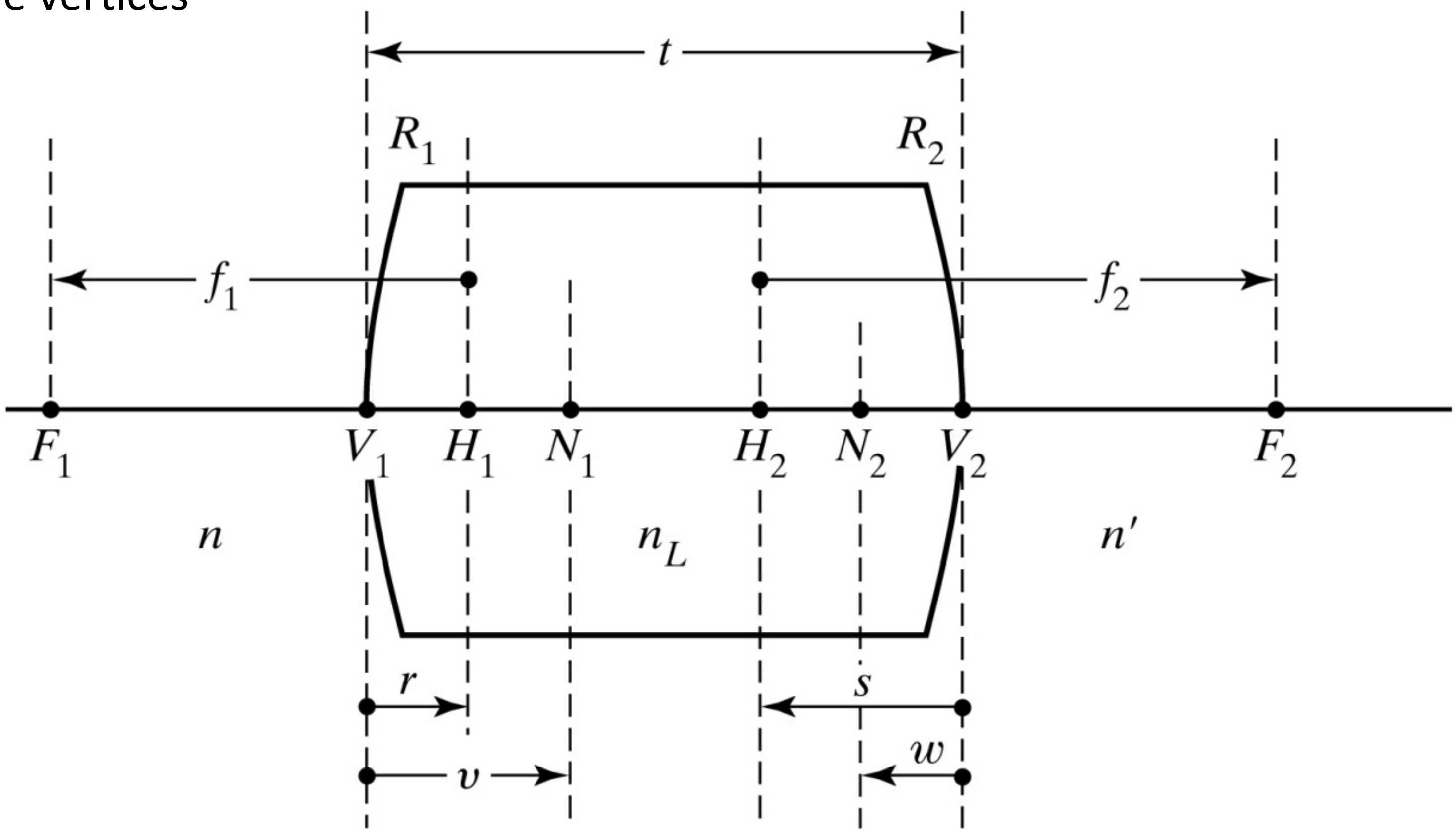
## Cardinal points and cardinal planes

**First and second nodal points** and **nodal planes.** Nodal points of a thick lens or any optical system permit correction to the ray that aims the center of the lens. Any ray that aims the first nodal point emerges from the second nodal point undeviated but slightly displaced.



## Cardinal points and cardinal planes

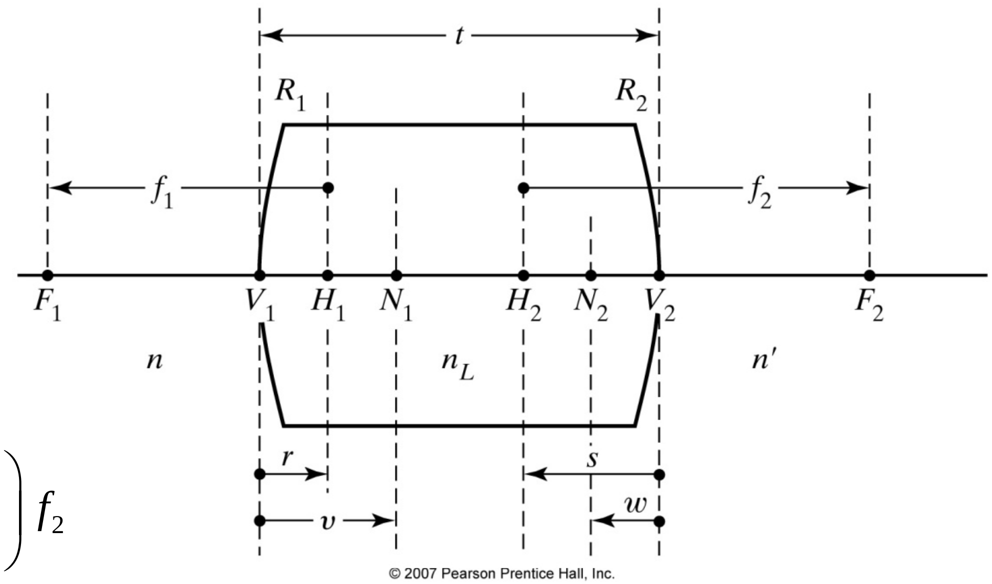
All the distances that are directed **to the left are negative (-)** and **directed to the right are positive (+)** by the sign convention. Notice that focal distances are not measured from the vertices



## Basic equations for the thick lens

$$\frac{1}{f_1} = \frac{n_L - n'}{nR_2} - \frac{n_L - n}{nR_1} - \frac{(n_L - n)(n_L - n')}{nn_L}\frac{t}{R_1R_2}$$

$$f_2 = -\frac{n'}{n} f_1 \qquad \text{for } n = n' \text{ then } f_2 = -f_1$$




Location of the principal planes:

$$r = \frac{n_L - n'}{n_L R_2} f_1 t; \quad s = \frac{n_L - n}{n_L R_1} f_2 t$$

The positions of the nodal points:

$$v = \left(1 - \frac{n'}{n} + \frac{n_L - n'}{n_L R_2} t\right) f_1; \quad \text{w} = \left(1 - \frac{n}{n'} + \frac{n_L - n}{n_L R_1} t\right) f_2$$

Image and object distances and lateral magnification:

$$-\frac{f_1}{s_o} + \frac{f_2}{s_i} = 1 \quad \text{and } m = -\frac{ns_i}{n' s_o}$$

The sign convention is as usual (real + and virtual –) as long as the distances are measured relative to their corresponding principla planes.

For an ordinary thin lens in air: $n = n' = 1$ and $r = v,\ s = w$ we arive at the usual thin lens equations:

$$\frac{1}{\text{s}_\text{o}} + \frac{1}{s_i} = \frac{1}{f} \quad \text{and } m = -\frac{s_i}{s_o} \text{ and } f = f_2 = -f_1$$

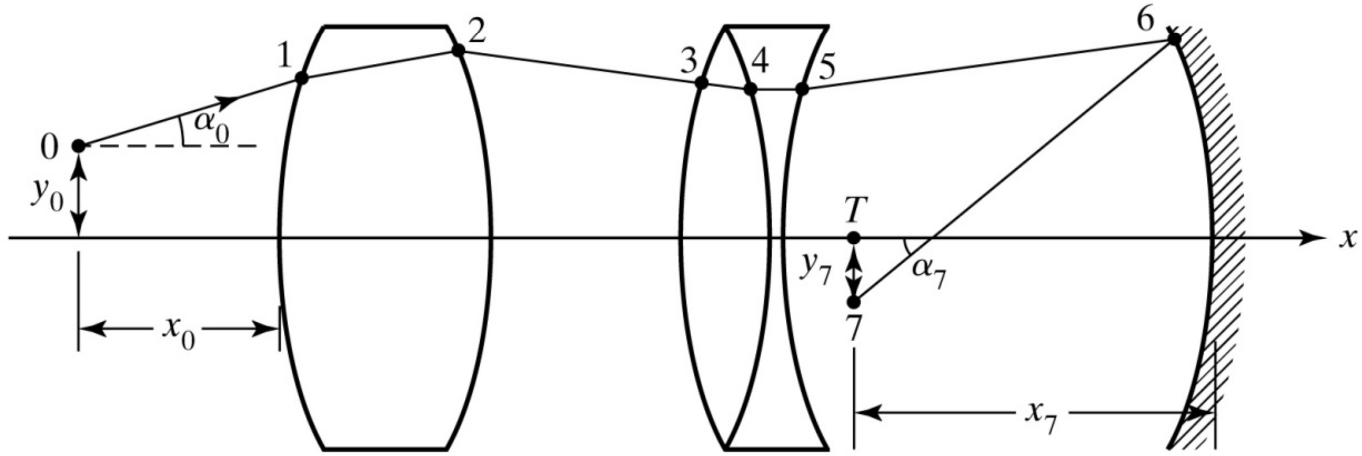
## The matrix methods in paraxial optics

For optical systems with many elements we use a systematic approach called matrix method.

We follow two parameters for each ray as it progresses through the optical system.

A ray is defined by its height and its direction (the angle it makes with the optical axis).

We can express $y_7$ and $\alpha_7$ in terms of $y_1$ and $\alpha_1$ multiplied by the transfer matrix of the system.



**The translational matrix**

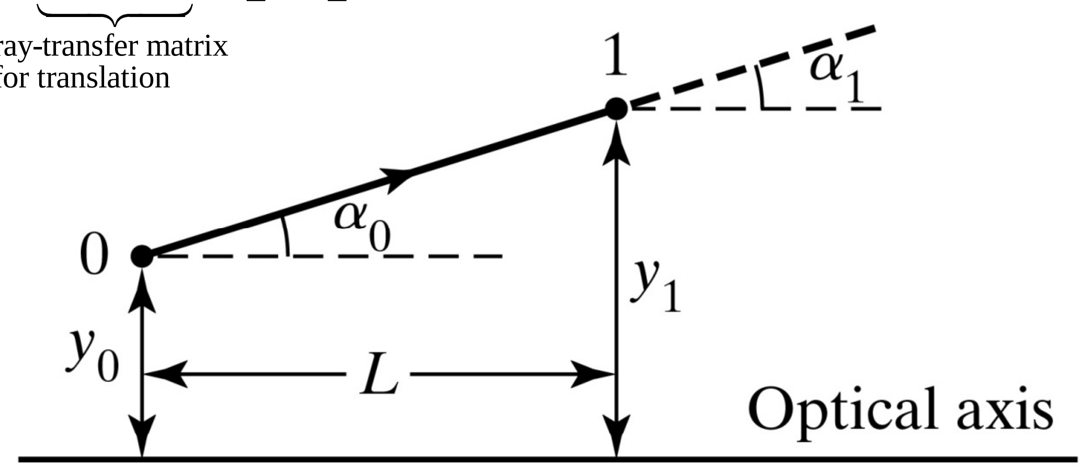
Consider simple tanslationof a ray in a homogeneous medium.

Translation from point 0 to 1 with paraxial approximation:

$\alpha_1 = \alpha_0$ and $y_1 = y_0 + L \tan \alpha_0 = y_0 + L\alpha_0$

We rewrite the equations:

$$\left.\begin{aligned} y_1 &= (1)\, y_0 + (L)\, \alpha_0 \\ \alpha_1 &= (0)\, y_0 + (1)\, \alpha_0 \end{aligned}\right\} \rightarrow \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}}_{\text{ray-transfer matrix for translation}} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$



## Refraction matrix

Consider refraction of a ray at a spherical interface (paraxial approximation):

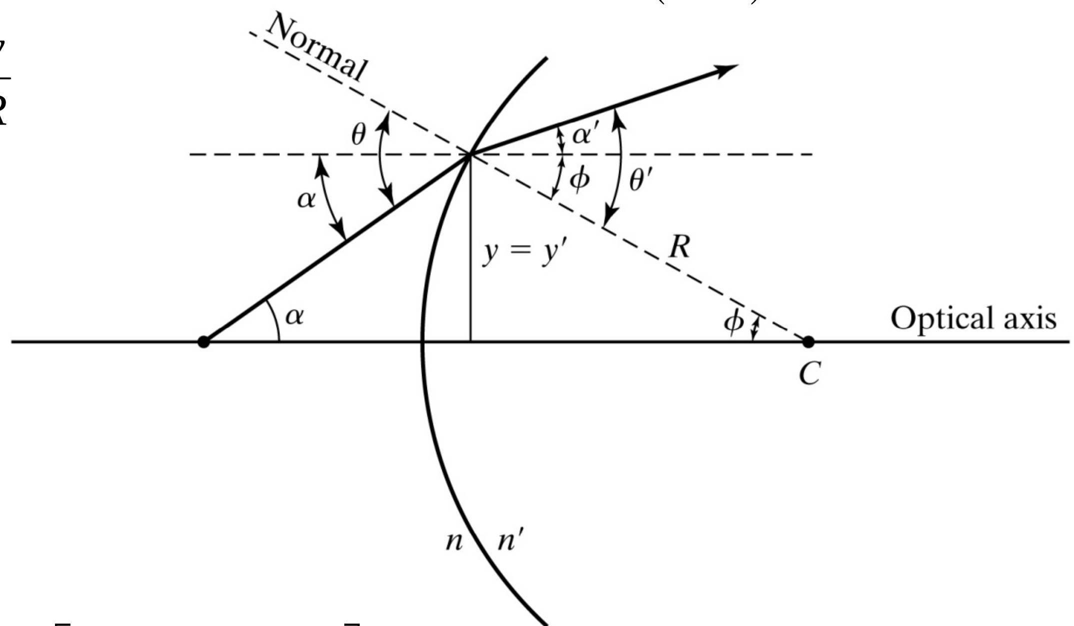
Ray coordinates before refraction $(\mathrm{y},\alpha)$ and ray coordinates after refraction $(\mathrm{y}',\alpha')$

$$\alpha' = \theta' - \phi = \theta' - \frac{y}{R} \text{ and } \alpha = \theta - \phi = \theta - \frac{y}{R}$$

Paraxial form of Snell's law: $n\theta = n'\theta'$

$$\alpha' = \left(\frac{n}{n'}\right)\theta - \frac{y}{R} = \left(\frac{n}{n'}\right)\left(\alpha + \frac{y}{R}\right) - \frac{y}{R}$$

$$\alpha' = \left(\frac{1}{R}\right)\left(\frac{n}{n'} - 1\right)y + \left(\frac{n}{n'}\right)\alpha$$





The approximate linear equations:

$$\left.\begin{aligned} \mathrm{y}' &= (1)y + (0)\alpha \\ \alpha' &= \left[\left(\frac{1}{R}\right)\left(\frac{n}{n'} - 1\right)\right]y + \left(\frac{n}{n'}\right)\alpha \end{aligned}\right\} \rightarrow \begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n}{n'} - 1\right) & \frac{n}{n'} \end{bmatrix}}_{\text{Ray-transfer matrix for refraction}} \begin{bmatrix} y \\ \alpha \end{bmatrix}$$

If $R \rightarrow \infty$ we have transfer matrix for refration by plane interface: $\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{bmatrix}}_{\text{Refraction by a plane}} \begin{bmatrix} y \\ \alpha \end{bmatrix}$

## The reflection matrix

Consider refraction of a ray at a spherical interface (paraxial approximation):

Ray coordinates before refraction $(\mathrm{y}, \alpha)$ and ray coordinates after refraction $(\mathrm{y'}, \alpha')$

$$\alpha' = \theta' - \phi = \theta' - \frac{y}{R} \text{ and } \alpha = \theta - \phi = \theta - \frac{y}{R}$$

Goal: connect $(\mathrm{y'}, \alpha')$ to $(\mathrm{y}, \alpha)$ by a ray transfer matrix for reflection by a concave mirror

Sign convention for the angles: $(+)$ pointing upward and $(-)$ pointing downward

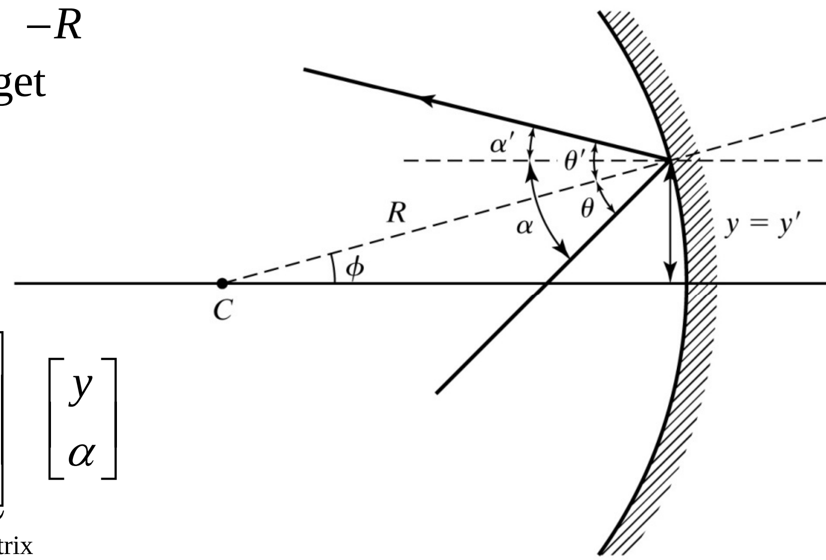
$$\alpha = \theta + \phi = \theta + \frac{y}{-R} \text{ and } \alpha' = \theta' - \phi = \theta' - \frac{y}{-R}$$

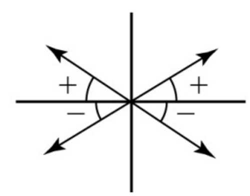
To eliminate $\theta$ and $\theta'$ we use $\theta = \theta'$ we get

$$\alpha' = \theta + \frac{y}{R} = \alpha + \frac{2y}{R}$$

The desired equations become:

$$\left.\begin{aligned} \mathrm{y'} &= (1)\,y + (0)\,\alpha \\ \alpha' &= \left(\frac{2}{R}\right) y + (1)\,\alpha \end{aligned}\right\} \rightarrow \begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}}_{\text{Ray-transfer matrix for reflection}} \begin{bmatrix} y \\ \alpha \end{bmatrix}$$



## The thick lens and thin lens matrices
## The general Ray-transfer matrix

Goal: construct a matrix that represents a thick lens with two different material on each side of it. In traversing the lens the ray undergoes two refractions and one translation for which we have derived the matrices. The radii of curvature are $(+)$ in this example. The symbolic equations are:

$$\begin{bmatrix} \mathrm{y}_1 \\ \alpha_1 \end{bmatrix} = M_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} \text{ for the first reflection}$$

$$\begin{bmatrix} \mathrm{y}_2 \\ \alpha_2 \end{bmatrix} = M_2 \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} \text{ for the translation}$$

$$\begin{bmatrix} \mathrm{y}_3 \\ \alpha_3 \end{bmatrix} = M_3 \begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} \text{ for the second reflectiont}$$

$$\rightarrow \begin{bmatrix} \mathrm{y}_3 \\ \alpha_3 \end{bmatrix} = \underbrace{M_3 M_2 M_1}_{\substack{M\text{: Transfer matrix} \\ \text{of the entire lens}}} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

The individual matrix operates on the ray in the same order in which the optical actins influense the ray.

No comutative propery for multiplication of matricies. Only associative property holds.

$$\mathrm{M}=M_3 M_2 M_1 = \left(M_3 M_2\right) M_1 = M_3 \left(M_2 M_1\right) \neq M_2 M_3 M_1$$

Generalizing the matrix relationship for any number of translating, reflecting, refracting surfaces:

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \underbrace{M_N M_{N-1} \cdots M_2 M_1}_{\substack{M\text{: Transfer matrix} \\ \text{of the entire lens}}} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} \text{ with } M = M_N M_{N-1} \cdots M_2 M_1 \text{ ray transfer matrix for the optical system.}$$

## The thick lens and thin lens matrices

Goal: Applying the results for a thick lens

Let R represent a reflaction matrix and T represent tsranslation

$M = R_2 T R_1$ the ray-transfer matrix for a thick lens can be written as:

$$M = \begin{bmatrix} 1 & 0 \\ \frac{n_L - n'}{n' R_2} & \frac{n_L}{n'} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{n - n_L}{n_L R_1} & \frac{n}{n_L} \end{bmatrix}$$

For a thin lens t $\rightarrow$ 0 in one environment $(n = n')$ the ray-transfer matrix becomes

$$M = \begin{bmatrix} 1 & 0 \\ \frac{n_L - n}{n R_2} & \frac{n_L}{n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{n - n_L}{n_L R_1} & \frac{n}{n_L} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_L - n}{n}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) & 1 \end{bmatrix}$$

We exprss the lower left hand element in terms of the focal length:

$$\frac{1}{f} = \frac{n_L - n}{n}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$ the lansmaker's formula

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the ray-transfer matrix for a thin lens also known as the ABCD matrix.

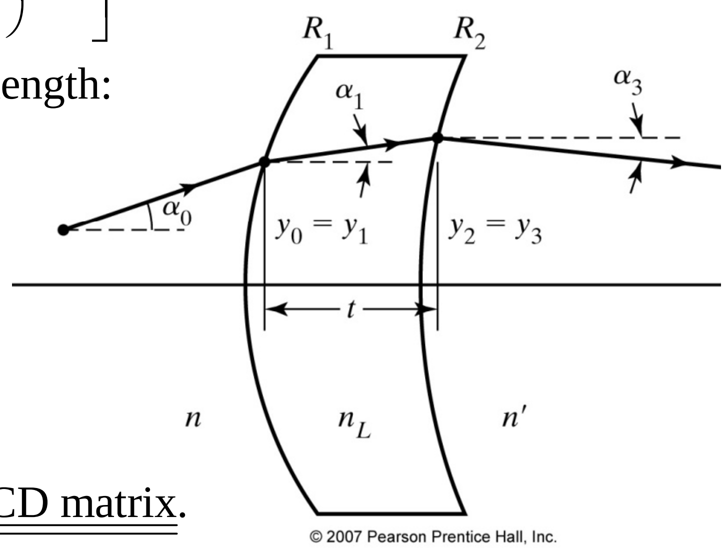
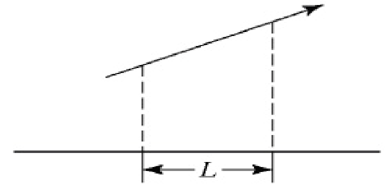


**TABLE 18-1** SUMMARY OF SOME SIMPLE RAY-TRANSFER MATRICES

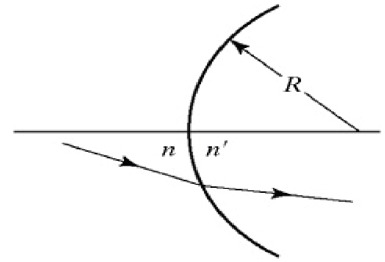
Translation matrix:

$$M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \mathfrak{T}$$




Refraction matrix, spherical interface:

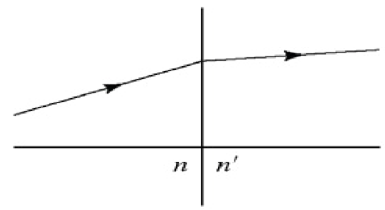
$$M = \begin{bmatrix} 1 & \overset{0}{\not{L}} \\ \dfrac{n - n'}{Rn'} & \dfrac{n}{n'} \end{bmatrix} = \mathfrak{R}$$




$(+R)$ : convex
$(-R)$ : concave

Refraction matrix, plane interface:

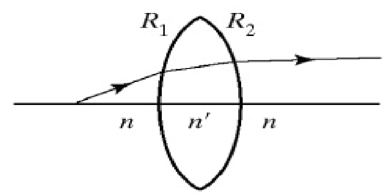
$$M = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n}{n'} \end{bmatrix}$$




Thin-lens matrix:

$$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

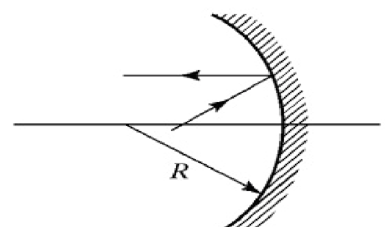
$$\frac{1}{f} = \frac{n' - n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$




$(+f)$ : convex
$(-f)$ : concave

Spherical mirror matrix:

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R} & 1 \end{bmatrix}$$




$(+R)$ : convex
$(-R)$ : concave

## Significance of system matrix elements

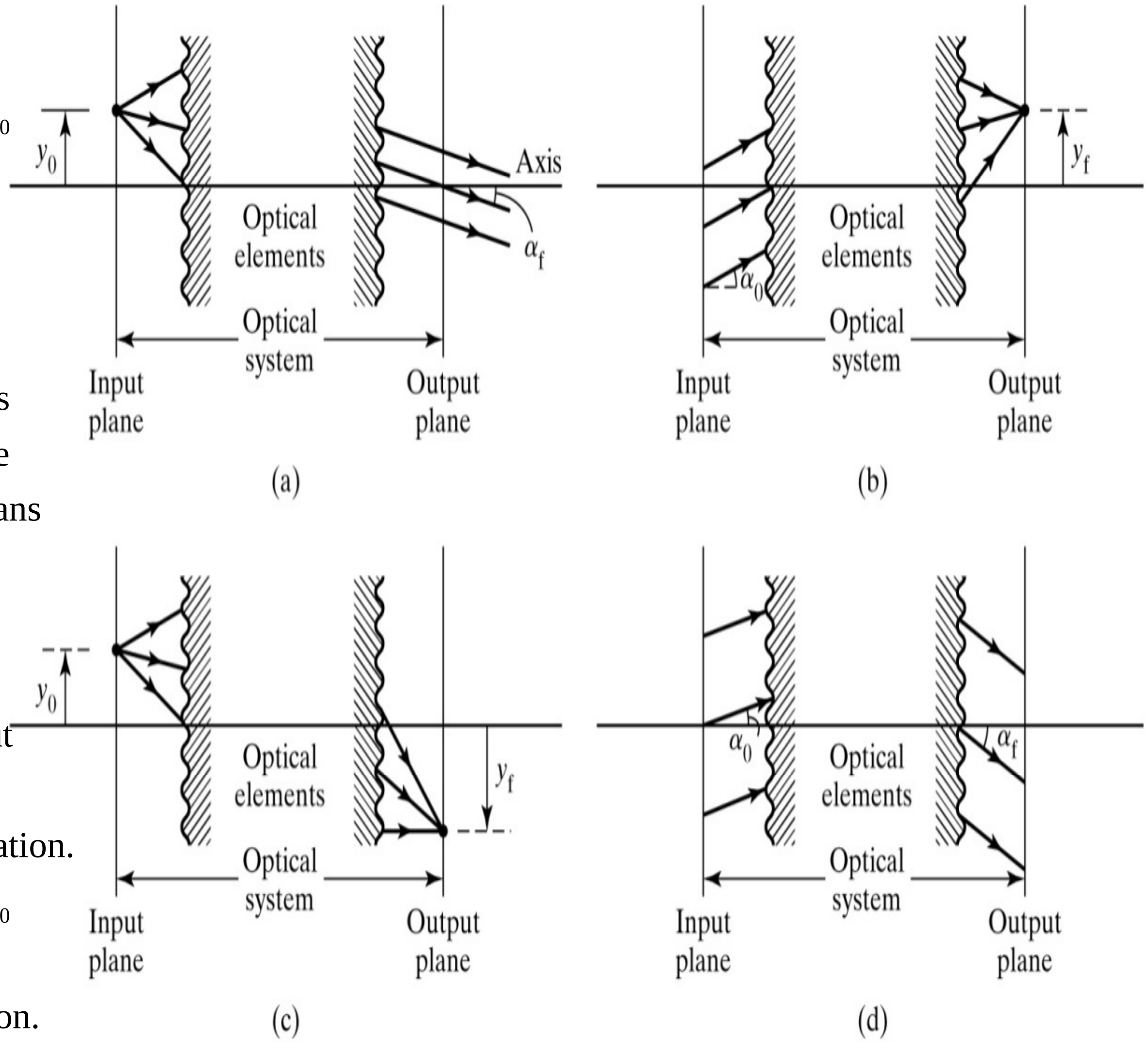
$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

a) If $D = 0 \rightarrow \alpha_f = Cy_0$ independent of $\alpha_0$
All the rays leavoing the input plane will have the same angle at the output plane. Input plane is on the first focal plane.

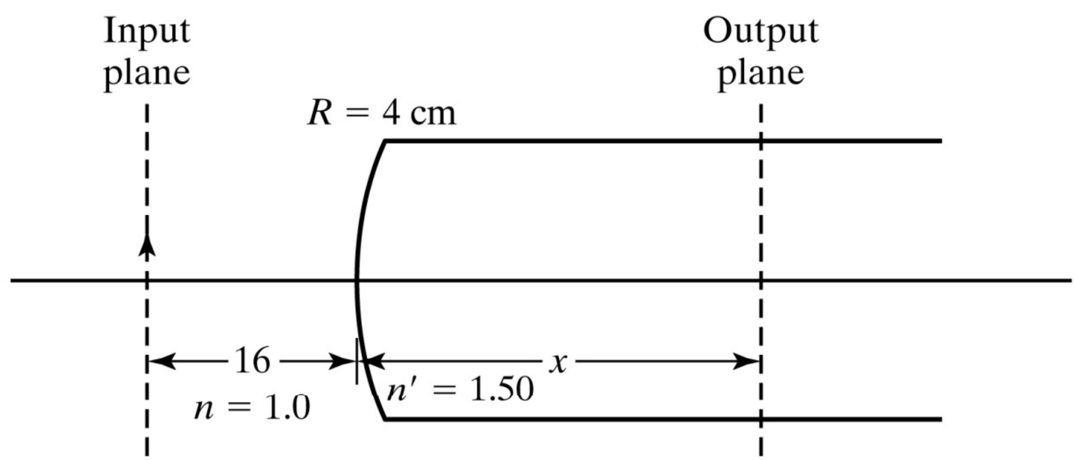
b) If $A = 0 \rightarrow y_f = B\alpha_0$
means $y_f$ is independent of $y_0$ that means all the rays departing input plane have the same height at the output plane. This means output plane is the second focal plane.

c) If $B = 0 \rightarrow y_f = Ay_0$ All the points leaving the input plane at hight $y_0$ will arrive the output plane at height $y_f$ output plane is image of the input plane.
A=$y_f / y_0$ corresponds to linear magnification.

d) If $C = 0 \rightarrow a_f = D\alpha_0$ independent of $y_0$
Input rays of all in one direction will produce output rays all in another direction. This is called (thelescopic system).



## Example 18.3



## Location of cardinal points for an optical system

Since the system ray-transfer matix explains the optical properties of an optical system we expect a relationship between the system matrix and location of the cardinal points.

Input and output planes define limits of an optical system.

We define distances locating six cardinal planes with respect to the input and output planes.

$F_1$ and $F_2$ are at $f_1$ and $f_2$ from the principal points at $H_1$ and $H_2$

$F_1$ and $F_2$ are at $p$ and $q$ from the reference input and output planes

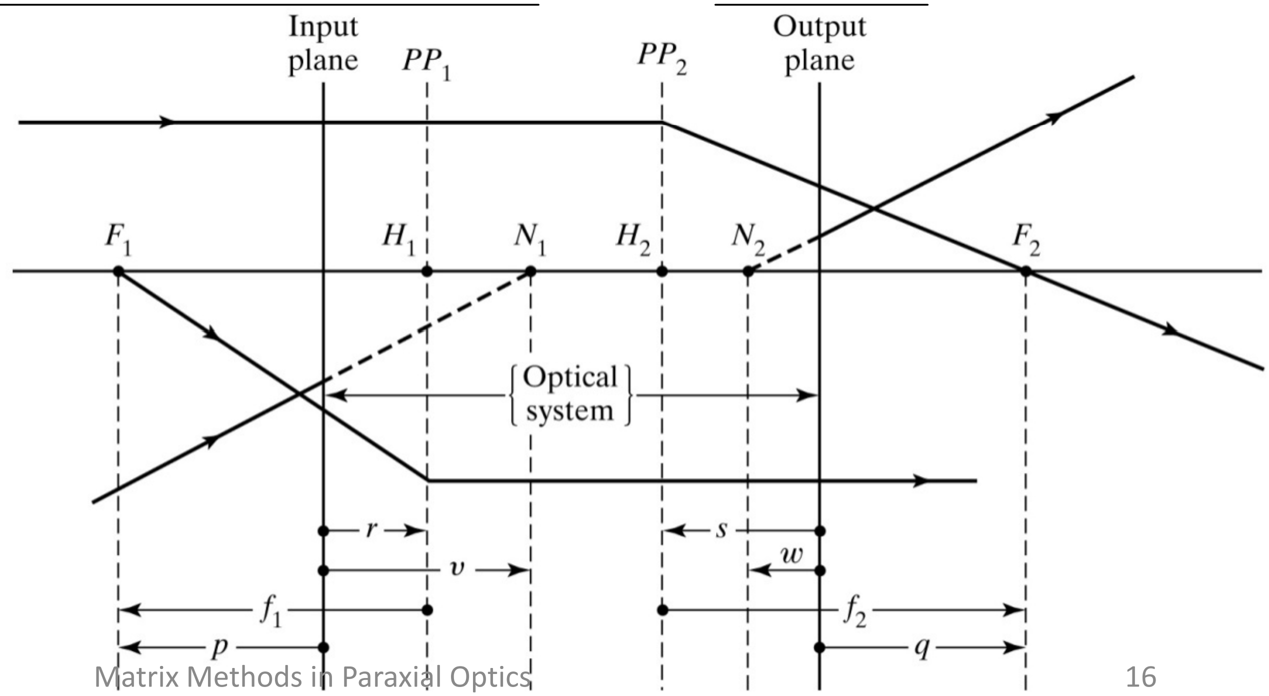
$r$ and $s$ are disances of the reference input and output planes from the principal points at $H_1$ and $H_2$

$v$ and $w$ are disances of the reference input and output planes from the nodal points at $N_1$ and $N_2$

Sign convention:

$(+)$ distance measured to the right of a reference plane

$(-)$ distance measured to the left of a reference plane



## Location of cardinal points

Input ray $(y_0, \alpha_0)$ and output ray $(y_f, 0)$ from figure (a)

$$\left.\begin{aligned} y_f &= Ay_0 + B\alpha_0 \\ 0 &= Cy_0 + D\alpha_0 \end{aligned}\right\} \rightarrow y_0 = -\left(\frac{D}{C}\right)\alpha_0$$

For small angles $\alpha_0 = \dfrac{y_0}{-p} \rightarrow \boxed{p = -\dfrac{y_0}{\alpha_0} = \dfrac{D}{C}}$

$p$ is negative that means it is to the left of the input reference plane.

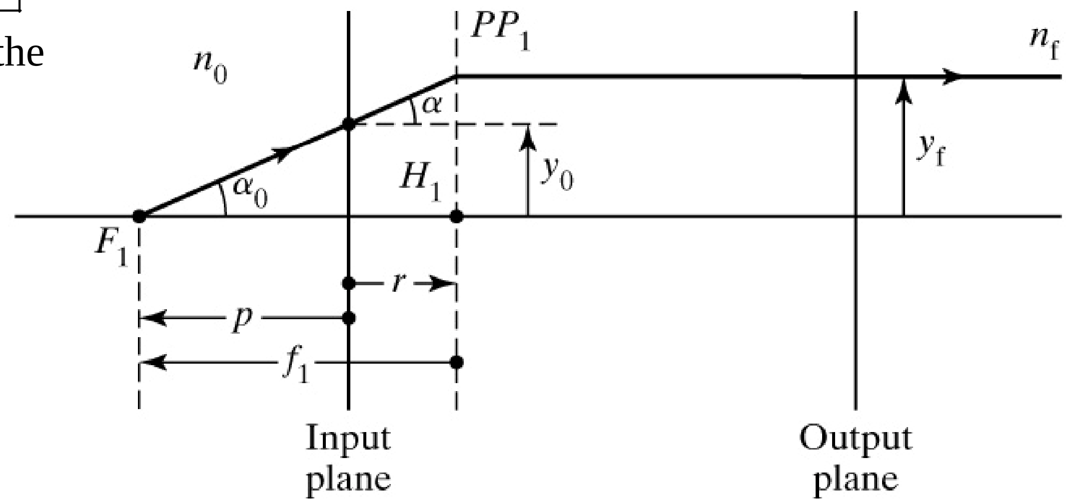
$$\alpha_0 = \frac{y_f}{-f_1}$$

$$f_1 = \frac{-y_f}{\alpha_0} = \frac{-(Ay_0 + B\alpha_0)}{\alpha_0} = \frac{AD}{C} - B$$

$$f_1 = \frac{AD - BC}{C} = \frac{Det(M)}{C} = \boxed{\left(\frac{n_0}{n_f}\right)\frac{1}{C} = f_1}$$

We used: $\boxed{Det(M) = AD - BC = \dfrac{n_0}{n_f}}$

$$r = p - f_1 = \boxed{\frac{1}{C}\left(D - \frac{n_0}{n_f}\right) = r}$$



(a)

## Location of cardinal points

Using figure (b) we can find $q,\ f_2, s$

$$q = -\frac{A}{C}; \quad s = \frac{1-A}{C}; \quad f_2 = q - s = -\frac{1}{C}$$

Using figure (c) we can find $v,\ w$

$$\alpha_0 = \alpha_f = \alpha = -\frac{y_0}{v}$$

Notice $y_0$ is negative i.e. below the optical axis

$$\alpha = Cy_0 + D\alpha \rightarrow \frac{y_0}{\alpha} = \frac{1-D}{C} = -v$$

$$v = \frac{D-1}{C} \quad \text{and} \quad w = \frac{(n_0/n_f) - A}{C}$$

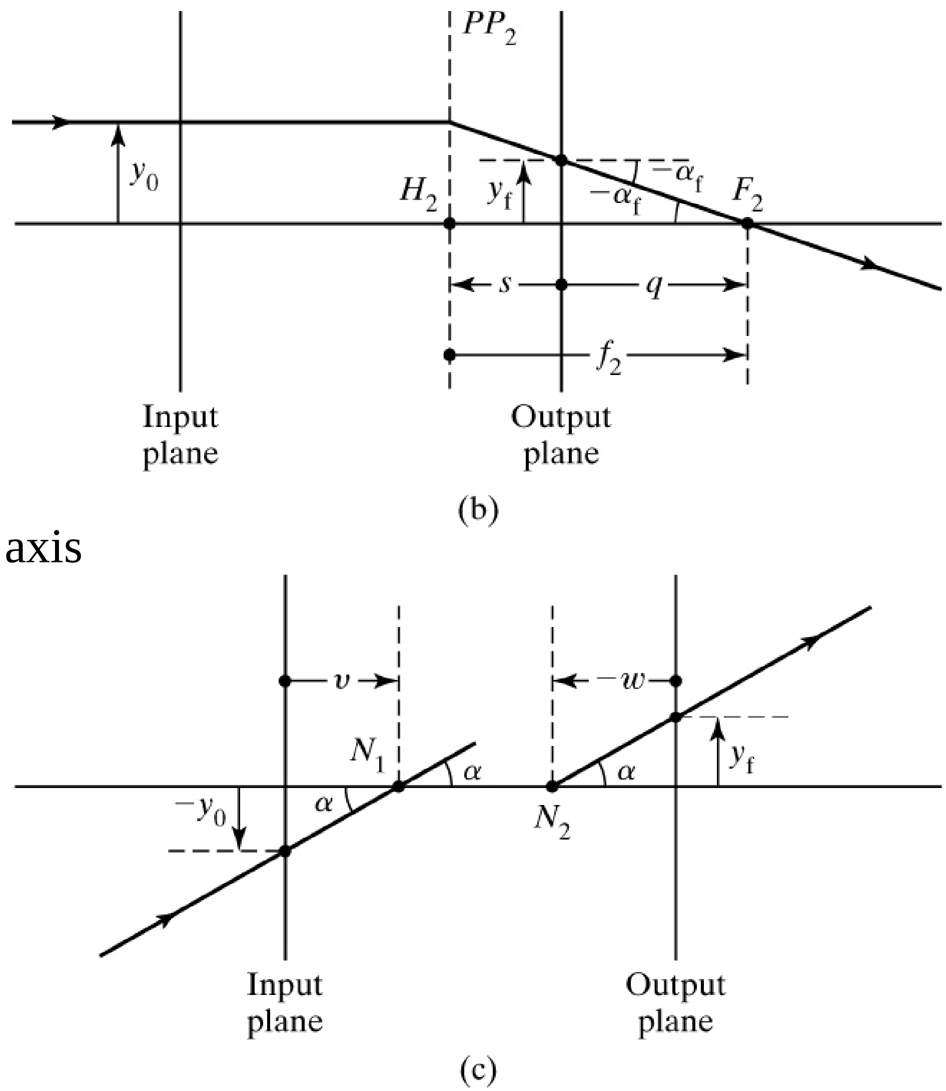


**TABLE 18-2** CARDINAL POINT LOCATIONS IN TERMS OF SYSTEM MATRIX ELEMENTS

| | | |
|---|---|---|
| $p = \dfrac{D}{C}$ | $F_1$ | Located relative to input (1) and output (2) reference planes |
| $q = -\dfrac{A}{C}$ | $F_2$ | |
| $r = \dfrac{D - n_0/n_f}{C}$ | $H_1$ | |
| $s = \dfrac{1 - A}{C}$ | $H_2$ | |
| $v = \dfrac{D - 1}{C}$ | $N_1$ | |
| $w = \dfrac{n_0/n_f - A}{C}$ | $N_2$ | |
| $f_1 = p - r = \dfrac{n_o/n_f}{C}$ | $F_1$ | Located relative to principal planes |
| $f_s = q - s = -\dfrac{1}{C}$ | $F_2$ | |

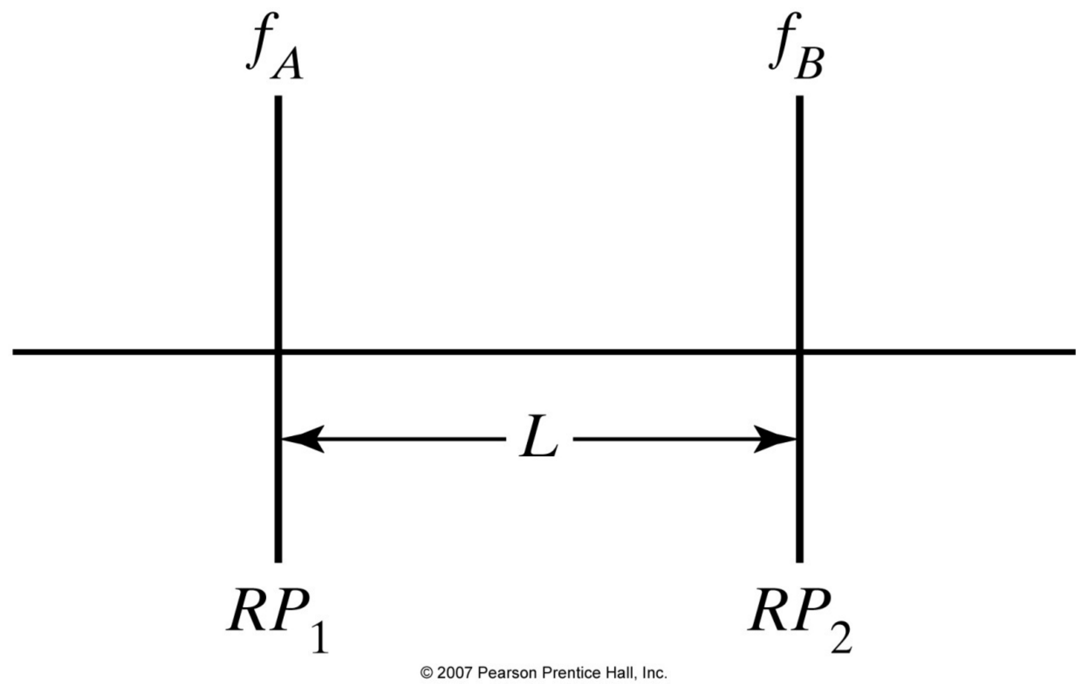


1) When initial and final material have the same index of refraction then $r = v$ and $s = w$ i.e. principal points and nodal points coinside

2) When initial and final material have the same index of refraction then first and second focal lengths are equal $f_1 = f_2$

3) The separation of the principal points is the same as separation of the nodal points or $r - s = v - w$

## Examples: tow thin lenses in air separated by a distance L



## Examples: tow thin lenses in air separated by a distance L

Focal lengths of the lenses $f_A$, $f_B$,

Assume the input and output reference planes are located on the lenses.

The system transfer matrix includes two thin-lens matrices and a translation matrix.

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_B} & 1 \end{bmatrix} \begin{bmatrix} 1 & \mathrm{L} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_B} & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{L}{f_A} & L \\ \frac{1}{f_B}\left(\frac{1}{f_A}-1\right)-\frac{1}{f_A} & 1-\frac{L}{f_B} \end{bmatrix}$$

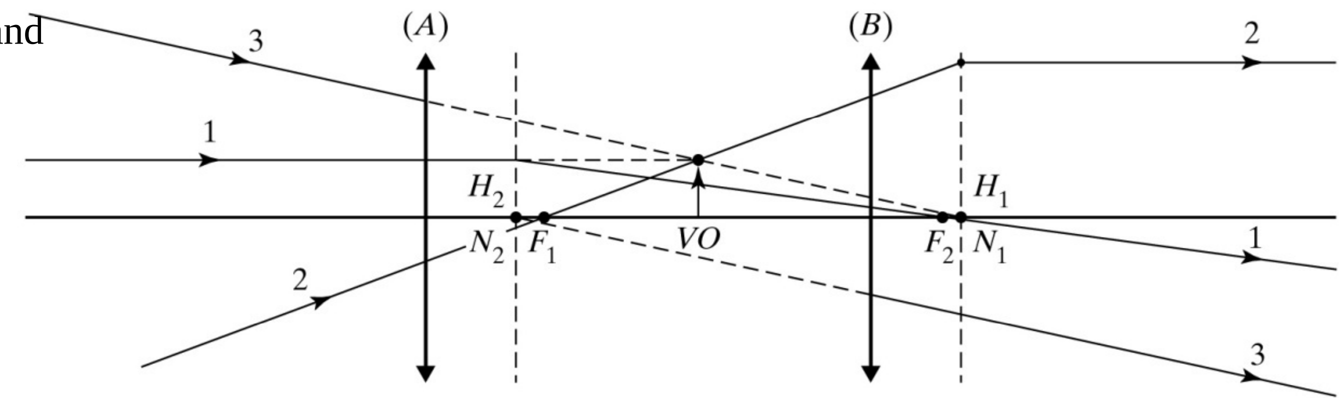
First focal length of the system: $f_1 = \frac{1}{C}$ and the second focal length of the system: $f_2 = -\frac{1}{C} = f_{eq}$

$$\frac{1}{f_{eq}} = \frac{1}{f_A} + \frac{1}{f_B} - \frac{L}{f_A f_B}$$

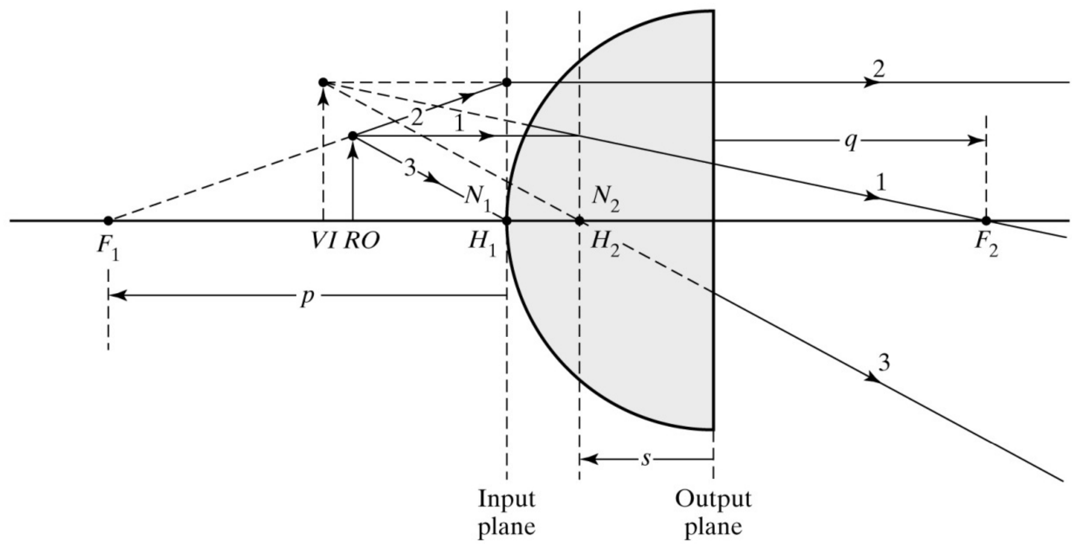
The first principal point and nodal point: $r = v = \left(\frac{f_{eq}}{f_B}\right)L$

The second principal point and

nodal point: $s = w = \left(\frac{f_{eq}}{f_A}\right)L$



## Example



## Ray tracing

Limiting analysis of optical systems to paraxial rays is an over simplification of the problem and ignores effect of aberrations.

**Ray tracing** is following the actual path of each ray through the system using laws of reflection and refraction. Traditionally it is done by hand and graphically but today it is all computerized.

We introduce a ray-tracing technique that is often limited to **meridional rays.**

**Meridional rays** are the rays that pass through the optical axis of the system.

Meridional rays tend to stay in the **meridional planes** as the laws of refraction/reflection require them.

This limits our treatment to a 2-dimensional space.

Skew rays are the ones that contribute to the image and do not pass the optical axis.

Analysis of the skew rays require a 3-dimensional treatment.

Understanding **aberrations require analysis of the non-paraxial rays and skew rays**.

Design of the complicated lens systems require knowledge and experience with ray-tracing techniques and optimizing performance of the system by changing system parameters and arriving at a perfect performance.

## Ray tracing

Goal: followin a meridional ray through a single spherical refracting surface.

$n,\ n'$: Indexes of refraction; $R$ radius of curvature; $A$ origin of the ray.

$\alpha, \alpha'$: angle with optical axis before and after refraction.

O point of intersection with the optical axis; $P$ with the refracting surface; $I$ with the optical axis after refraction.

$I$ & $O$ are conjugate poits with distances $s, s'$ from the vertices.

Q: perpendicular distance from the vertex, V, to the incident ray.

$\theta, \theta'$: angles of incidence and refraction.

Sign convention: distances to the left of vertex $-$ and to the right of $V$ are $+$ above the optical axis $+$ and below are $-$.

From left to right, the angles have the same sign as the slopes.

Input parameters for each ray: $h$: elevation, $\alpha$: angle, and

$D$: distance from the vertex parallel to the optical axis.

From figure we write:

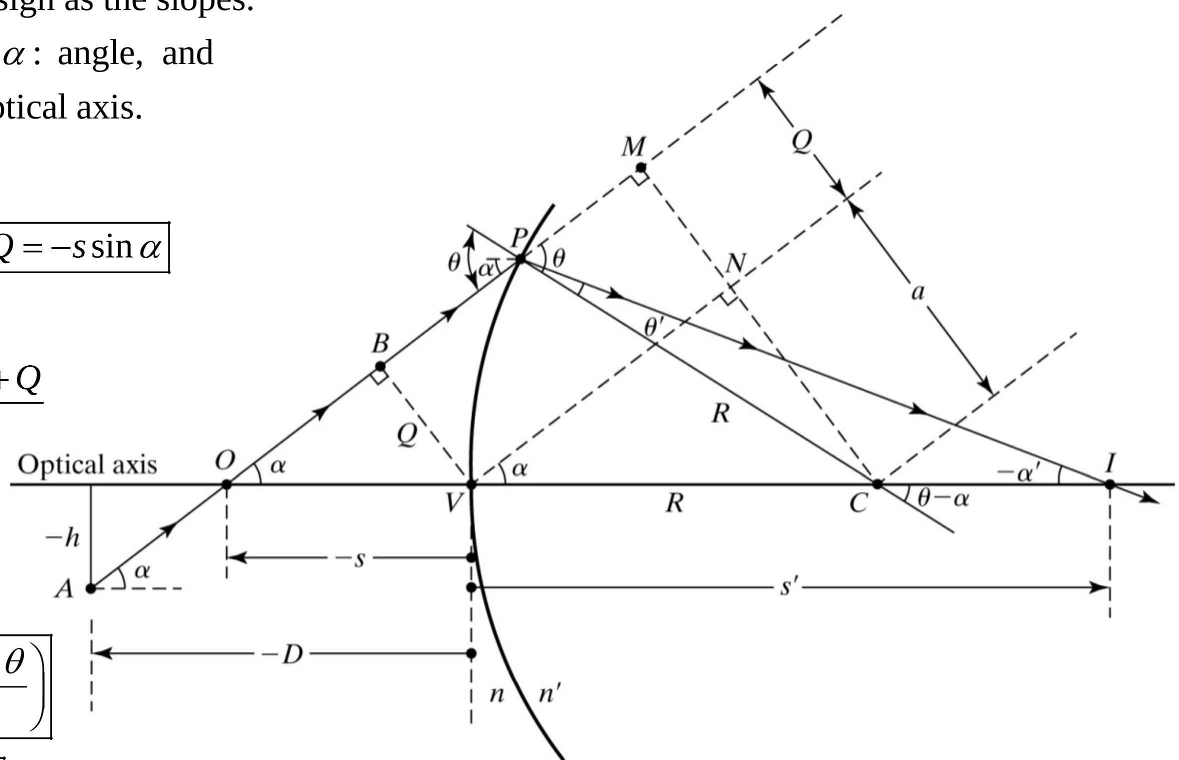
$$\boxed{s = D - \frac{h}{\tan\alpha}};\ \text{in } \Delta OBV \rightarrow \sin\alpha = \frac{Q}{-s} \rightarrow \boxed{Q = -s\sin\alpha}$$

$$\left.\begin{aligned} &\text{in } \Delta PMC \rightarrow \sin\theta = \frac{a+Q}{R} \\ &\text{in } \Delta VNC \rightarrow \sin\alpha = \frac{a}{R} \end{aligned}\right\} \sin\theta = \frac{R\sin\alpha + Q}{R}$$

$$\boxed{\theta = \sin^{-1}\left(\frac{Q}{R} + \sin\alpha\right)}$$

$$\text{At P} \rightarrow n\sin\theta = n'\sin\theta' \rightarrow \boxed{\theta' = \sin^{-1}\left(\frac{n\sin\theta}{n'}\right)}$$

$$\text{in } \Delta CPI \rightarrow \theta - \alpha = \theta' - \alpha' \rightarrow \boxed{\alpha' = \theta' - \theta + \alpha}$$



## Ray tracing

Q': perpendicular distance from the vertex, V, to the refracted ray.

$$\left.\begin{aligned}\text{In } \Delta CMV \rightarrow \sin(-\alpha') &= \frac{a'}{R} \\ \text{In } \Delta PLC \rightarrow \sin(\theta') &= \frac{Q'-a'}{R}\end{aligned}\right\} \boxed{Q' = R(\sin\theta' - \sin\alpha')}$$

$$\text{In } \Delta ITV \rightarrow \sin(-\alpha') = \frac{Q'}{s'} \rightarrow \boxed{s' = \frac{-Q'}{\sin\alpha'}}$$

Now we have the new values for the refracted ray $\alpha', Q', s'$ which prepare us for the next refraction in the sequence. But before that we need to calculate effect of the transfer by $t$ in the material with index $n'$.

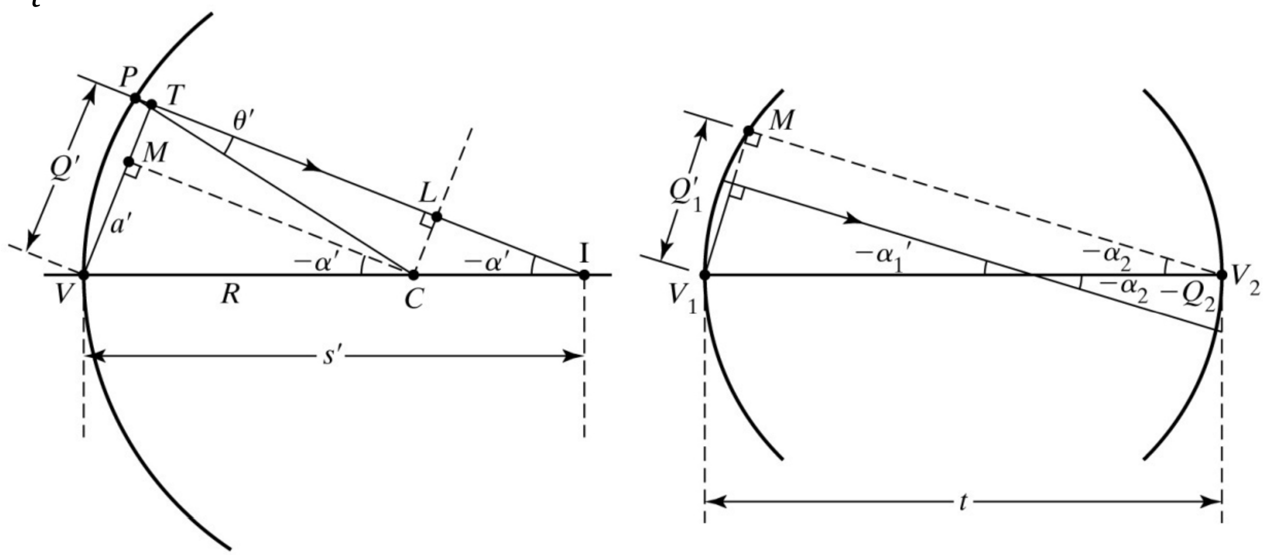
$$\text{In } \Delta V_2MV_1 \rightarrow \sin(-\alpha_2) = \frac{V_1M}{t} = \frac{Q'_1 - Q_2}{t} \rightarrow \boxed{Q_2 = Q'_1 + t\sin\alpha_2}$$

And $\boxed{\alpha_2 = \alpha'}$

We need to modify the equations for the special cases:

1) Incident ray is parallel to the optical axis.

2) Surface is plane with an infinite radius of curvature.




(a) (b)

## TABLE 18-3 MERIDIONAL RAY-TRACING EQUATIONS (INPUT: $n, n', R, \boldsymbol{\alpha}, h, D$)

| General case | Ray parallel to axis: $\alpha = 0$ | Plane surface: $R \Rightarrow \infty$ |
|---|---|---|
| $s = D - \dfrac{h}{\tan \alpha}$ | — | $s = D - \dfrac{h}{\tan \alpha}$ |
| $Q = -s \sin \alpha$ | $Q = h$ | $Q = -s \sin \alpha$ |
| $\theta = \sin^{-1}\left(\dfrac{Q}{R} + \sin \alpha\right)$ | $\theta = \sin^{-1}\left(\dfrac{Q}{R} + \sin \alpha\right)$ | — |
| $\theta' = \sin^{-1}\left(\dfrac{n \sin \theta}{n'}\right)$ | $\theta' = \sin^{-1}\left(\dfrac{n \sin \theta}{n'}\right)$ | — |
| $\alpha' = \theta' - \theta + \alpha$ | $\alpha' = \theta' - \theta + \alpha$ | $\alpha' = \sin^{-1} \dfrac{n}{n' \sin \alpha}$ |
| $Q' = R(\sin \theta' - \sin \alpha')$ | $Q' = R(\sin \theta' - \sin \alpha')$ | $Q' = Q \dfrac{\cos \alpha'}{\cos \alpha}$ |
| $s' = \dfrac{-Q'}{\sin \alpha'}$ | $s' = \dfrac{-Q'}{\sin \alpha'}$ | $s' = \dfrac{-Q'}{\sin \alpha'}$ |

*Transfer:* Input: $t$

$Q = Q' + t \sin \alpha'$

$\alpha = \alpha'$

$n = n'$

*Input:* new $n'$, $R$

*Return:* to calculate $\theta$

## Example ray tracing

Do a ray trace for two rays through a rapid landscape photographic lens of three elements. The parallel rays enter the lens from a distant object at altitudes of 1 and 5 mm above the optical axis. The lens specifications are:

$R_1 = -120.8$

$R_2 = -34.6 \quad t_1 = 6 \quad n_1 = 1.521$

$R_3 = -96.2 \quad t_2 = 2 \quad n_2 = 1.581$

$R_4 = -51.2 \quad t_3 = 3 \quad n_3 = 1.514$

The rays are parallel to the axis so we use the second column of the table

| Input | Results ray at h=1 | Results ray at h=5 |
|---|---|---|
| *First surface*: | | |
| $n = 1, n' = 1.521$ | $Q = 1$ | $Q = 5$ |
| $\alpha = 0$ | $\alpha' = 0.1625^0$ | $\alpha' = 0.8128^0$ |
| $h = 1$ or $5$ | $s' = -352.66$ | $s' = -352.53$ |
| $R$ = -120.8 | $Q' = 1.0000$ | $Q' = 5.0010$ |
| *Second surface*: | | |
| $t = 6$ | $Q = 1.0170$ | $Q = 5.0861$ |
| $n = 1.581$ | $\alpha' = 0.2202^0$ | $\alpha' = 1.1041^0$ |
| $R$ = -34.6 | $s' = -264.59$ | $s' = -264.03$ |
| | $Q' = 1.0170$ | $Q' = 5.0876$ |
| *Third surface*: | | |
| $t = 2$ | $Q = 1.0247$ | $Q = 5.1261$ |
| $n = 1.514$ | $\alpha' = 0.2030^0$ | $\alpha' = 1.0178^0$ |
| $R$ = -96.2 | $s' = -289.26$ | $s' = -288.58$ |
| | $Q' = 1.0247$ | $Q' = 5.1260$ |
| *Final surface*: | | |
| $t = 3$ | $Q = 1.0353$ | $Q = 5.1793$ |
| $n = 1.581$ | $\alpha' = -0.2883^0$ | $\alpha' = -1.4520^0$ |
| $R$ = -51.2 | $\boxed{s' = -205.72}$ | $\boxed{s' = -203.91}$ |
| | $Q' = 1.0353$ | $Q' = 5.1672$ |

There is no common focus $\Delta s' = 1.8mm$