Chapter 14
Matrix Treatment of Polarization

Lecture Notes for Modern Optics based on
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Spring 2009

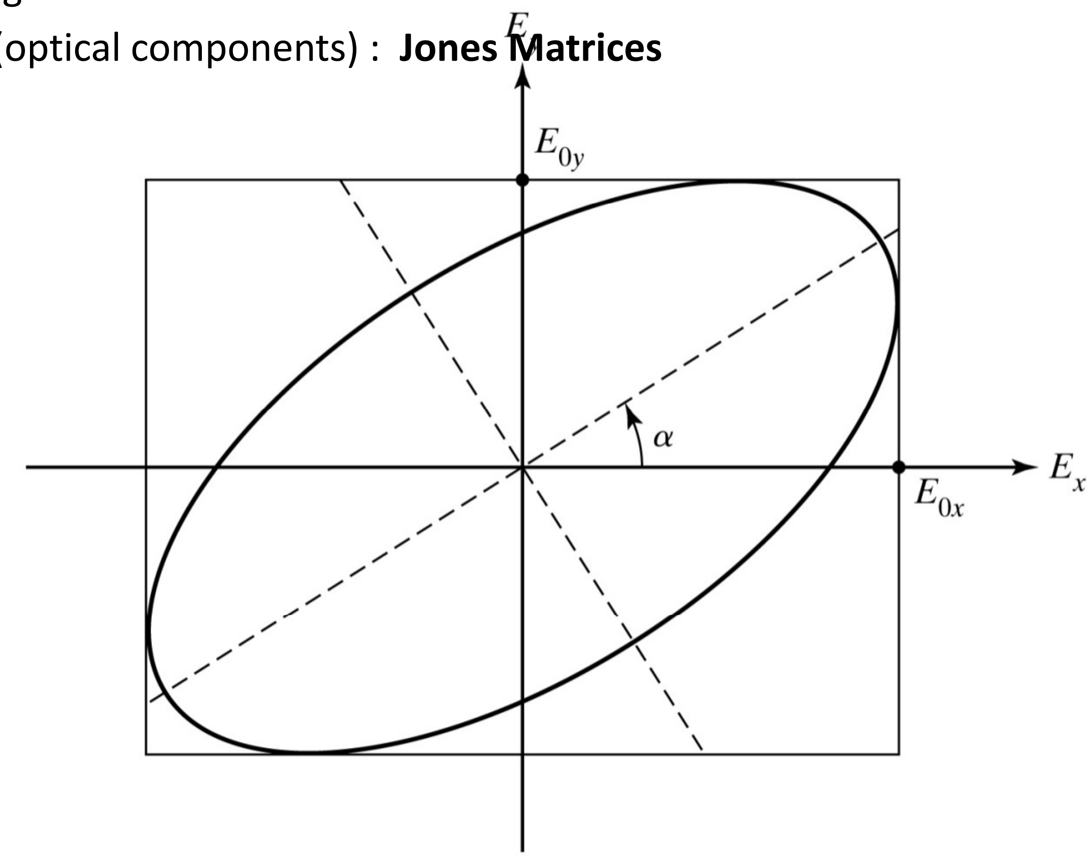
Polarization

Polarization of an electromagnetic wave is direction of the electric field vector **E.**

Mathematical presentation of polarized light: **Jones Vectors**

Mathematical presentation of polarizers (optical components) : **Jones Matrices**

- **Linear polarizer**
- **Phase retarder**
- **Rotators**



Mathematical presentation of polarized light

Electric field of an electromagnetic wave propagating along z-direction:

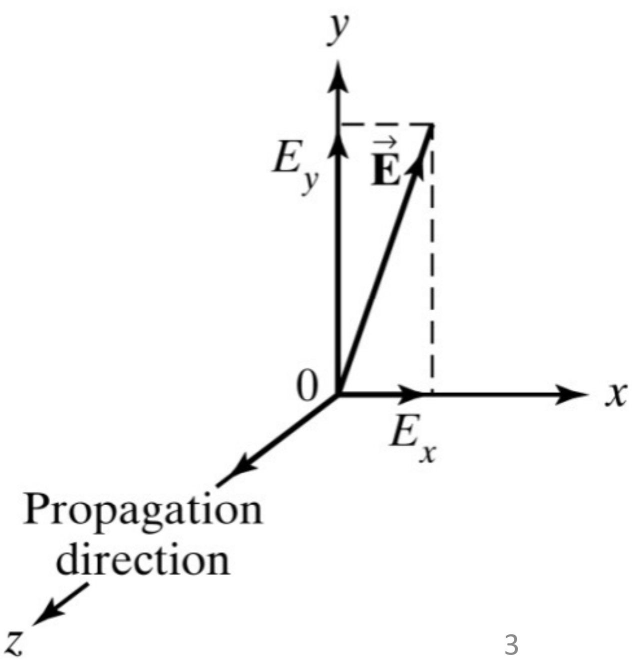
$$\vec{\mathbf{E}} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} \text{ with } \begin{cases} \tilde{E}_x = E_{0x} e^{i(kz-\omega t+\phi_x)} \\ \tilde{E}_y = E_{0y} e^{i(kz-\omega t+\phi_y)} \end{cases} \text{ and complex field components } \begin{cases} E_x = \mathrm{Re}(\tilde{E}_x) \\ E_y = \mathrm{Re}(\tilde{E}_y) \end{cases}$$

$$\tilde{\mathbf{E}} = E_{0x} e^{i(kz-\omega t+\phi_x)} \hat{\mathbf{x}} + E_{0y} e^{i(kz-\omega t+\phi_y)} \hat{\mathbf{y}} = \underbrace{\left[E_{0x} e^{i\phi_x} \hat{\mathbf{x}} + E_{0y} e^{i\phi_y} \hat{\mathbf{y}} \right]}_{\text{Complex amplitude vector } \tilde{\mathbf{E}}_0 \text{ also contains phase}} \underbrace{e^{i(kz-\omega t)}}_{\text{Plane wave}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$$

State of polarization of a wave is determined by relative amplitudes and phases of of the components of $\tilde{\mathbf{E}}_0$ that constitute a vector dubbed Jones vector:

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \end{bmatrix} = \begin{bmatrix} E_{0x} e^{i\phi_x} \\ E_{0y} e^{i\phi_y} \end{bmatrix}$$

Jones vector are normalized if $(E_{0x})^2 + (E_{0y})^2 = 1$.



Special cases of Jones vector

Particular forms of Jones vector:

$\phi_x = \phi_y = 0$

linearly polarized light along a line making an angle α with the x axis:

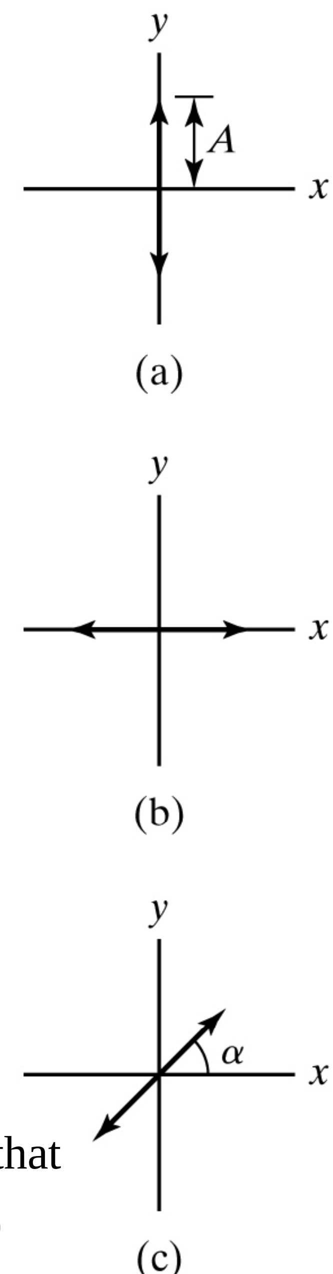
$$\tilde{E}_0 = \begin{bmatrix} E_{0x}e^{i\phi_x} \\ E_{0y}e^{i\phi_y} \end{bmatrix} = \begin{bmatrix} A\cos\alpha \\ A\sin\alpha \end{bmatrix} = A\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

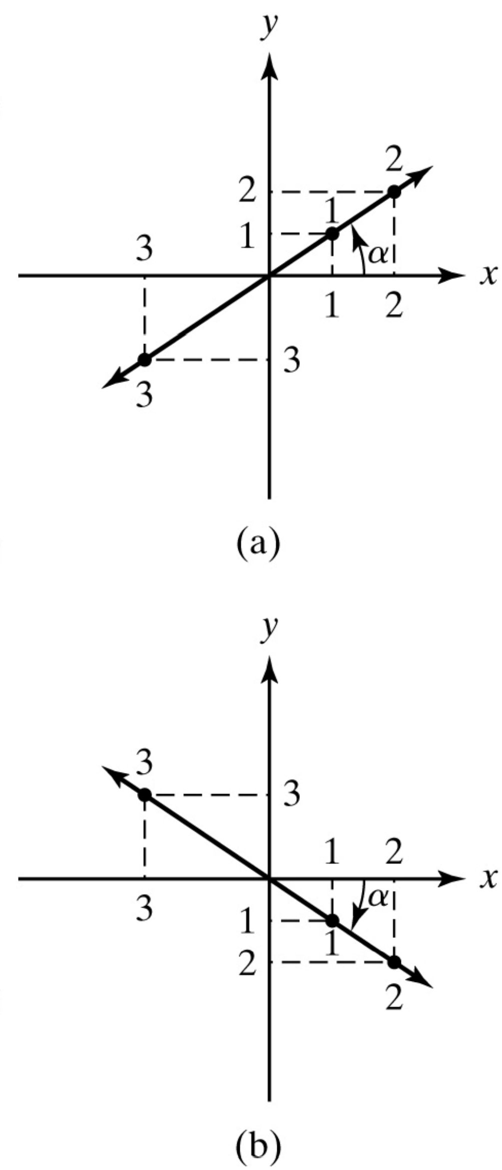
Vertical polarization: $\tilde{E}_0 = \begin{bmatrix} \cos(\pi/2) \\ \sin(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Horizontal polarization: $\tilde{E}_0 = \begin{bmatrix} \cos(0) \\ \sin(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Polarized at $\alpha = 60^0$: $\tilde{\mathbf{E}}_0 = \begin{bmatrix} \cos(60^0) \\ \sin(60^0) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

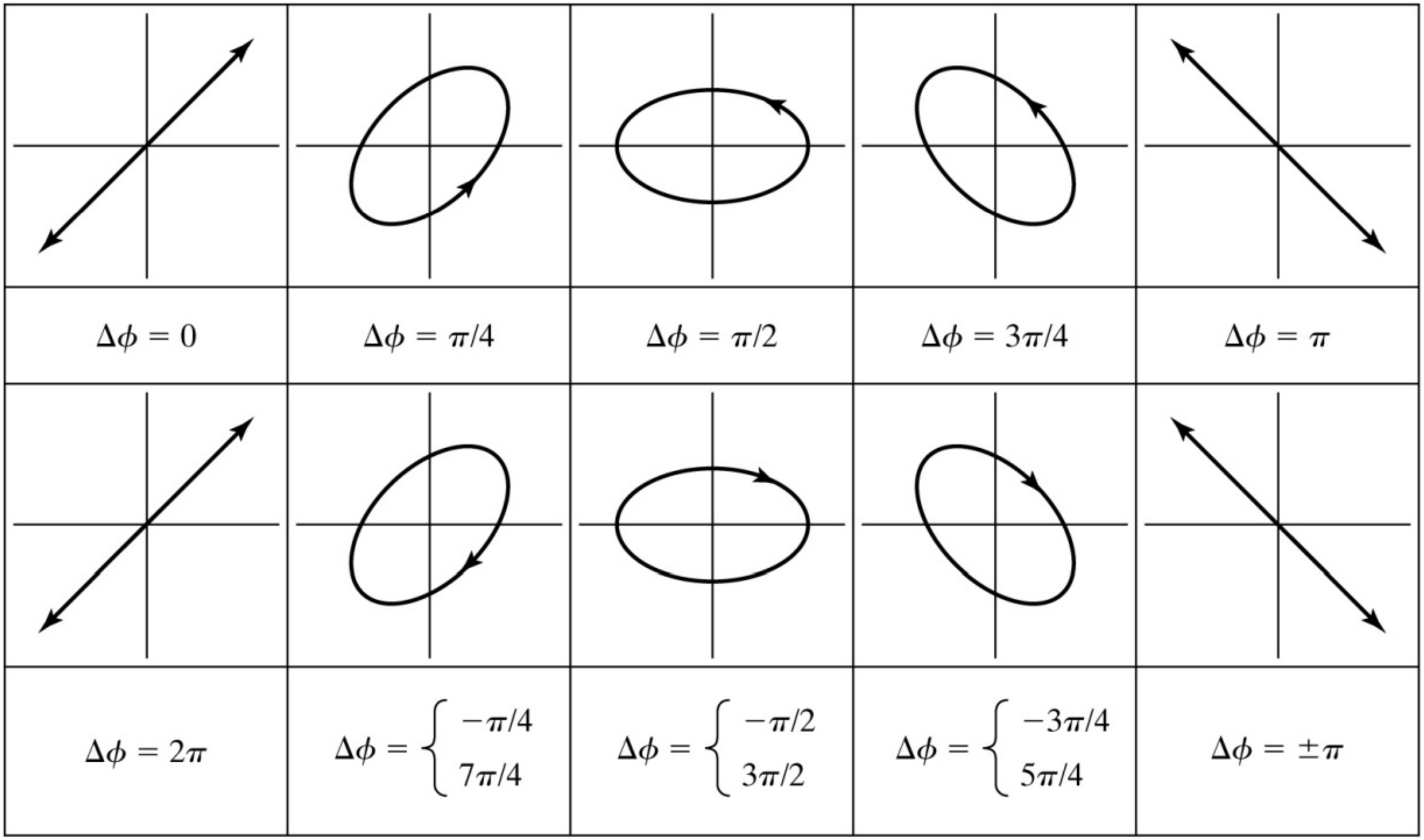
Conclusion1: The light presented by a Jones vactor that both of its elements, a and b, are real (not both zero) is a linearly polarized light along the angle $\boxed{\alpha = \tan^{-1}(b/a)}$





Lissajous Figures

For general case of $\phi_x \neq 0$ and $\phi_y \neq 0$ the head of the $\vec{\mathbf{E}}$ vector traces an ellipse rather than a straight line. The relative phase difference of the E_{ox} and E_{oy}, $\Delta\phi = \phi_y - \phi_x$ determines the shape of the Lissajous figure and the state of polarization of the wave.



LCP and RCP

Example: consider electric field of an EM wave that has $E_{0x} = E_{oy} = A$ and E_x leads E_y by $\varepsilon = \pi/2$. Determine the state of polarization and deduce the normalized Jones vectors for this light.

We write the complex amplitudes as

$$\begin{cases} \tilde{E}_x = E_{0x} e^{-i\omega t} \\ \tilde{E}_y = E_{0y} e^{-i(\omega t - \varepsilon)} \end{cases} \rightarrow \begin{cases} E_x = A\cos\omega t \\ E_y = A\cos(\omega t - \varepsilon) \end{cases} \rightarrow \begin{cases} E_x = A\cos\omega t \\ E_y = A\cos\left(\omega t - \dfrac{\pi}{2}\right) \end{cases} \rightarrow \begin{cases} E_x = A\cos\omega t \\ E_y = A\sin\omega t \end{cases}$$

$E^2 = E_x^2 + E_y^2 = A^2\left(\cos^2\omega t + \sin^2\omega t\right) = A^2$ the **E** vector traces out a circle of radius A.

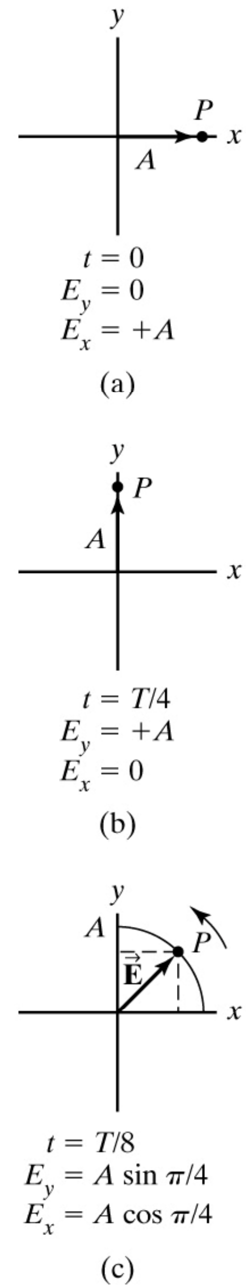
Finding the Jones vector: $\begin{cases} E_{0x} = E_{oy} = A \\ \phi_x = 0,\ \phi_y = \pi/2 \end{cases}$ then $\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x} e^{i\phi_x} \\ E_{oy} e^{i\phi_y} \end{bmatrix} = \begin{bmatrix} A \\ Ae^{i\pi/2} \end{bmatrix} = A\begin{bmatrix} 1 \\ i \end{bmatrix}$

Normalization: $\tilde{E}_0 \tilde{E}_0^* = 1 \rightarrow A^2\left(1^2 + \left(ii^*\right)\right) = 2A^2 = 1 \rightarrow A = 1/\sqrt{2}$

The normalized Jones vector is: $\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$ We call this a left-circularly polarized light or LCP since when we view this light head-on we see the $\mathbf{E_0}$ vector tip is rotating counterclockwise on a circle of radius $1/\sqrt{2}$. Figure shows the $\mathbf{E_0}$ at diffrent times.

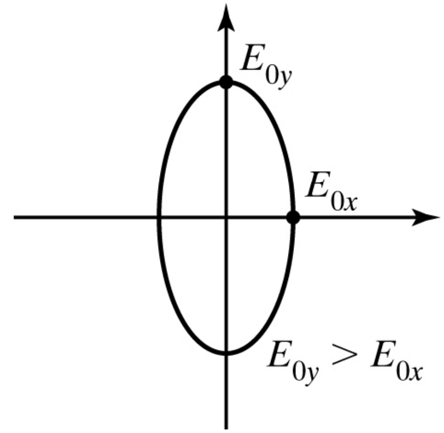
If E_y leads E_x by $\pi/2$ the $\mathbf{E_0}$ would rotate clockwise. $\tilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

We have right-circularly polarized light or RCP in this case.



Elliptically polarized light

Example: consider electric field of an EM wave that has $E_{0x} = A$ and $E_{oy} = B$ where A and B are positive numbers and $\underline{E_x \text{ leads/lags } E_y \text{ by } \varepsilon = \pi/2}$. Determine the state of polarization and deduce the normalized Jones vectors for this light.



(a)

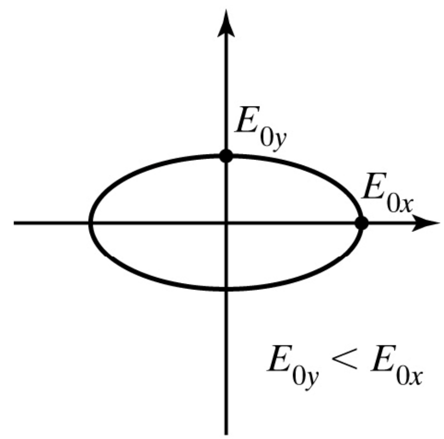
$$\begin{cases} \widetilde{E}_x = E_{0x} e^{-i\omega t} \\ \widetilde{E}_y = E_{0y} e^{-i(\omega t-\varepsilon)} \end{cases} \rightarrow \begin{cases} E_x = A\cos\omega t \\ E_y = B\cos(\omega t-\varepsilon) \end{cases} \rightarrow \begin{cases} E_x = A\cos\omega t \\ E_y = B\cos\left(\omega t-\dfrac{\pi}{2}\right) \end{cases}$$

$$\rightarrow \begin{cases} E_x = A\cos\omega t \\ E_y = B\sin\omega t \end{cases}$$

Normalization: $\widetilde{E}_0 \widetilde{E}_0^* = 1 \rightarrow \left(A^2 + \left(iB(iB)^*\right)\right) = A^2 + B^2 = 1$

Jones vector counterclockwise $\widetilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{A^2+B^2}} \begin{bmatrix} A \\ Be^{i\pi/2} \end{bmatrix} = \dfrac{1}{\sqrt{A^2+B^2}} \begin{bmatrix} A \\ iB \end{bmatrix}$

Jones vector clockwise $\widetilde{\mathbf{E}}_0 = \dfrac{1}{\sqrt{A^2+B^2}} \begin{bmatrix} A \\ Be^{-i\pi/2} \end{bmatrix} = \dfrac{1}{\sqrt{A^2+B^2}} \begin{bmatrix} A \\ -iB \end{bmatrix}$



(b)

Conclusion 2: the Jones vector with elements un-equal in magnitude, one of which is pure imaginary, represents an elliptically polarized light.

Figure shows the $\mathbf{E_0}$ for two cases of $E_{0y} > E_{0x}$ (major axis along y) and $E_{0y} < E_{0x}$ (major axis along x).

Elliptically polarized light oriented at an angle relative to x-axis

Example: consider electric field of an EM wave that has $E_{0x} = A$ and $E_{oy} = b$ where A and B are positive numbers and E_x and E_y have phase difference of $\Delta\phi \neq \pm(m+1/2)\pi$ and $\Delta\phi \neq \pm m\pi$ where $m = 0, \pm 1, \pm 2, \ldots$

Determine the state of polarization and deduce the normalized Jones vectors for this light.

We can assume $\Delta\phi = \varepsilon$ and $\phi_x = 0,\ \phi_y = \varepsilon$

$$\tilde{\mathbf{E}}_0 = \begin{bmatrix} E_{0x}e^{i\phi_x} \\ E_{0y}e^{i\phi_y} \end{bmatrix} = \begin{bmatrix} A \\ be^{i\varepsilon} \end{bmatrix} = \begin{bmatrix} A \\ b\cos\varepsilon + ib\sin\varepsilon \end{bmatrix} = \begin{bmatrix} A \\ B+iC \end{bmatrix}$$ Counterclockwise rotation, general case

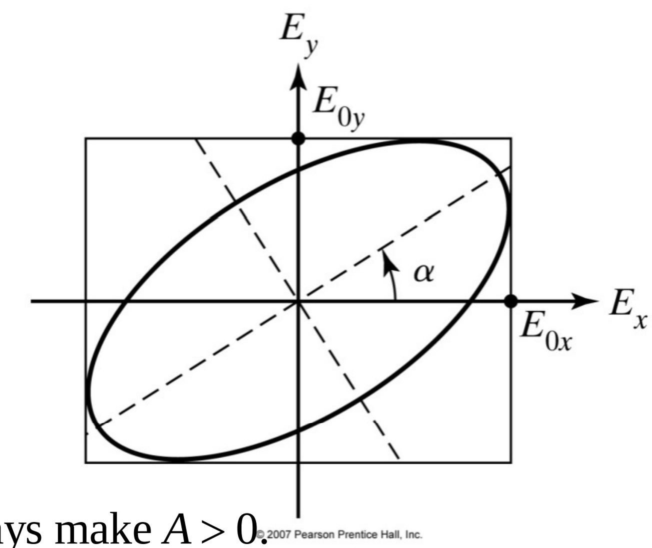
Normalization: $\tilde{E}_0 \tilde{E}_0^* = 1 \rightarrow \left(A^2 + \left((B+iC)(B+iC)^*\right)\right) = A^2 + B^2 + C^2 = 1$

Jones vector of an elliptically polarized light with major axis inclined at an angle α is:

$$\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{A^2+B^2+C^2}} \begin{bmatrix} A \\ B+iC \end{bmatrix} \quad \text{where } \tan 2\alpha = \frac{2E_{0x}E_{oy}\cos\varepsilon}{E_{0x}^2 - E_{0y}^2}$$

and $E_{0x} = A,\ E_{0y} = \sqrt{B^2+C^2},\ \varepsilon = \tan^{-1}\left(\frac{C}{B}\right)$

If $A > 0$ and $\begin{cases} C > 0 \text{ counteclockwise} \\ C < 0 \text{ clockwise} \end{cases}$



Note: polarization state represented by the Jones vector does not change if it is multiplied by a constant. So we can always make $A > 0$.

Usefulness and some properties of the Jones vectors

Two properties of the polarization vector or Jones vector :

1) polarization state of a wave does not change if its Jones vector is multiplied by a constant.

It only affects the amplitude.

2) polarization state of a wave does not change if its Jones vector is multiplied by a constant phase factor $e^{i\phi}$.

It promotes phase of each element by ϕ but not the phase difference $\Delta\phi$.

Example 1: illustrating usefulness of the Jones vectors.

a) Polarization state of a superposition of two waves can be found by adding the Jones vectors:

$$\begin{bmatrix} 1 \\ i \end{bmatrix} + \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Conclusion: We can generate a linearly polarized light with mixing equal portions of LCP and RCP light.

Example 2: superposition of horizontally and vertically linearly polarized light:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

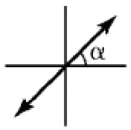
Conclusion : *b*y mixing equal protions of vertically and horizontally linearly polarized light we can get linearly polarized light at an angle 45^0.

Summary of the polarization states and their Jones vectors

TABLE 14-1 SUMMARY OF JONES VECTORS $\tilde{E}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix}$

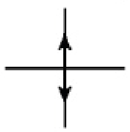
I. Linear Polarization ($\Delta\varphi = m\pi$)

General:

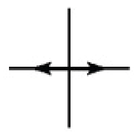


$\tilde{\mathbf{E}}_0 = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$

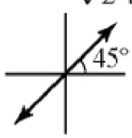
Vertical: $\tilde{\mathbf{E}}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



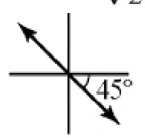
Horizontal: $\tilde{\mathbf{E}}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



At + 45°: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

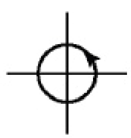


At −45°: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$



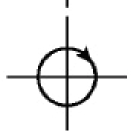
II. Circular Polarization $\left(\Delta\phi = \frac{\pi}{2}\right)$

Left:



$\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

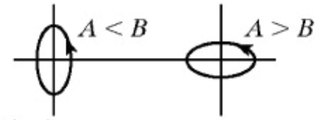
Right:



$\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

III. Elliptical Polarization

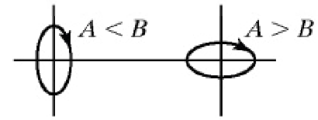
Left:



$\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{A^2 + B^2}}\begin{bmatrix} A \\ iB \end{bmatrix}$ $A > 0, B > 0$

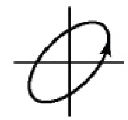
($\Delta\phi = (m + 1/2)\,\pi$)

Right:



$\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{A^2 + B^2}}\begin{bmatrix} A \\ -iB \end{bmatrix}$ $A > 0, B > 0$

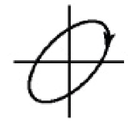
Left:



$\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{A^2 + B^2 + C^2}}\begin{bmatrix} A \\ B + iC \end{bmatrix}$ $A > 0, C > 0$

$\left(\Delta\phi \neq \begin{cases} m\pi \\ (m + 1/2)\pi \end{cases}\right)$

Right:



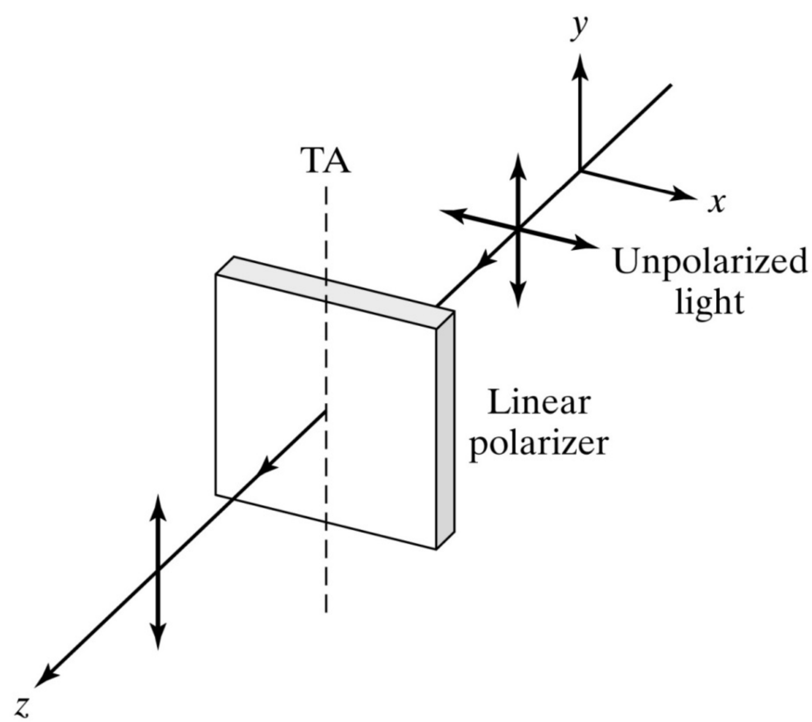
$\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{A^2 + B^2 + C^2}}\begin{bmatrix} A \\ B - iC \end{bmatrix}$ $A > 0, C > 0$

Mathematical presentation of polarizers

We can represent an optical instrument or device by its transfer matrix or abcd matrix: $\mathbf{M}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

There are optical devices that affect (change) state of polarization of the light. Our goal is to represent each of these devices with a transfer matrix such that multiplying it with the Jones vector of the original light produces the resulting light.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} E_{0x}e^{i\phi_x} \\ E_{oy}e^{i\phi_y} \end{bmatrix}=\begin{bmatrix} E_x e^{i\phi_x} \\ E_y e^{i\phi_y} \end{bmatrix} \text{ or } \mathbf{M}\widetilde{\mathbf{E}}_{\mathbf{0}}=\widetilde{\mathbf{E}}$$



Linnear polarizers

Linear polarizer: it selectively removes virations in a given direction and transmits in perpendicular direction.

Partial polarization: sometimes the process of removing other polarizations is partial and not 100% efficient. See the figure how the output light is polarized along the transmission axis (TA).

Along the TA: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{cases} a(0)+b(1)=0 \\ c(0)+d(1)=1 \end{cases}$

Perpendicular to TA: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} a(1)+b(0)=0 \\ c(1)+d(0)=0 \end{cases}$

The result for linear polarizer along the y axis (vertically) is:

Linear polarizer, TA vertical $\rightarrow \mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Linear polarizer, TA horizontal $\rightarrow \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

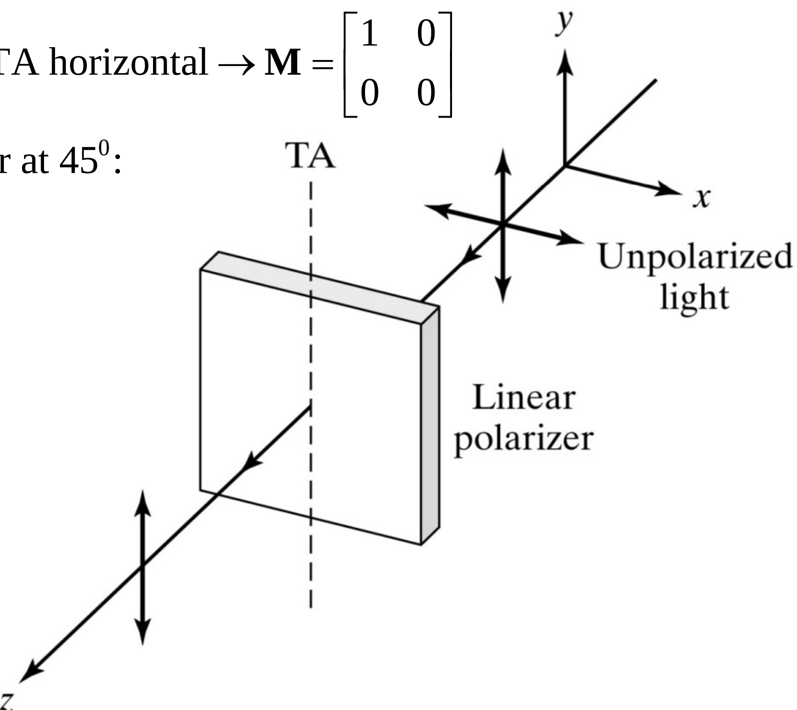
Exercise: Derive the polarization matices for the linear polarizer at 45^0:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{Polarization same as the polarizer's TA}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{Polarization perpendicular to the polarizer's TA}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear polarizer, TA at 45^0: $\mathbf{M} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

The most general case of the linear polarizer:

Linear polarizer, TA at θ: $\mathbf{M} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$



Phase retarder

The pase retarder introduces a phase difference between the orthogonal polarization components. If the speed of light in each orthogonal direction is different, there would be a cumulative phase difference $\Delta\phi$ between the components as light emerges from the media.

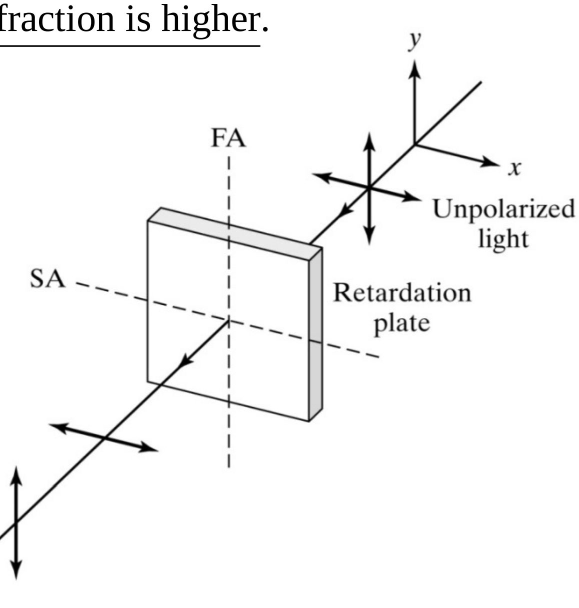
Fast axis (FA): the axis along which the speed of light is faster or index of refraction is lower.

Slow axis (SA): the axis along which the speed of light is slower or index of refraction is higher.

Finding the matrix for retarder: we want a matrix that will transform

$E_{0x}e^{i\phi_x}$ to $E_{0x}e^{i(\phi_x+\varepsilon_x)}$ and $E_{0y}e^{i\phi_y}$ to $E_{0y}e^{i(\phi_y+\varepsilon_y)}$

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{M}}\begin{bmatrix} E_{0x}e^{i\phi_x} \\ E_{0y}e^{i\phi_y} \end{bmatrix}=\begin{bmatrix} E_{0x}e^{i(\phi_x+\varepsilon_x)} \\ E_{0y}e^{i(\phi_y+\varepsilon_y)} \end{bmatrix}\rightarrow \mathbf{M}=\begin{bmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{bmatrix} \text{ Phase retarder}$$



Quarter – wave plate (QWP), a retarder with the net phase difference $\pi/2$

$$\begin{cases} \varepsilon_x-\varepsilon_y=\dfrac{\pi}{2} \text{ SA horizontal, let } \varepsilon_x=\dfrac{\pi}{4}, \text{ and } \varepsilon_y=-\dfrac{\pi}{4}\rightarrow \mathbf{M}=\begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} \\ \varepsilon_y-\varepsilon_x=\dfrac{\pi}{2} \text{ SA vertical, let } \varepsilon_x=-\dfrac{\pi}{4}, \text{ and } \varepsilon_y=+\dfrac{\pi}{4}\rightarrow \mathbf{M}=\begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \end{cases}$$

$$\mathbf{M}=e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\text{QWP, SA vertical,} \quad \mathbf{M}=e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\text{QWP, SA horizontal}$$

Half – wave plate (HWP): a retarder with the net phase difference $|\varepsilon_x-\varepsilon_y|=\pi$

$$\mathbf{M}=e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\text{QWP, SA vertical,} \quad \mathbf{M}=e^{i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\text{QWP, SA horizontal}$$

Phase rotator

The phase rotator rotates the drection of polarization of the linearly polarized light by some angle β.

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta+\beta) \\ \sin(\theta+\beta) \end{bmatrix} \rightarrow$$

$$\mathbf{M} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \text{ Phase rotator}$$

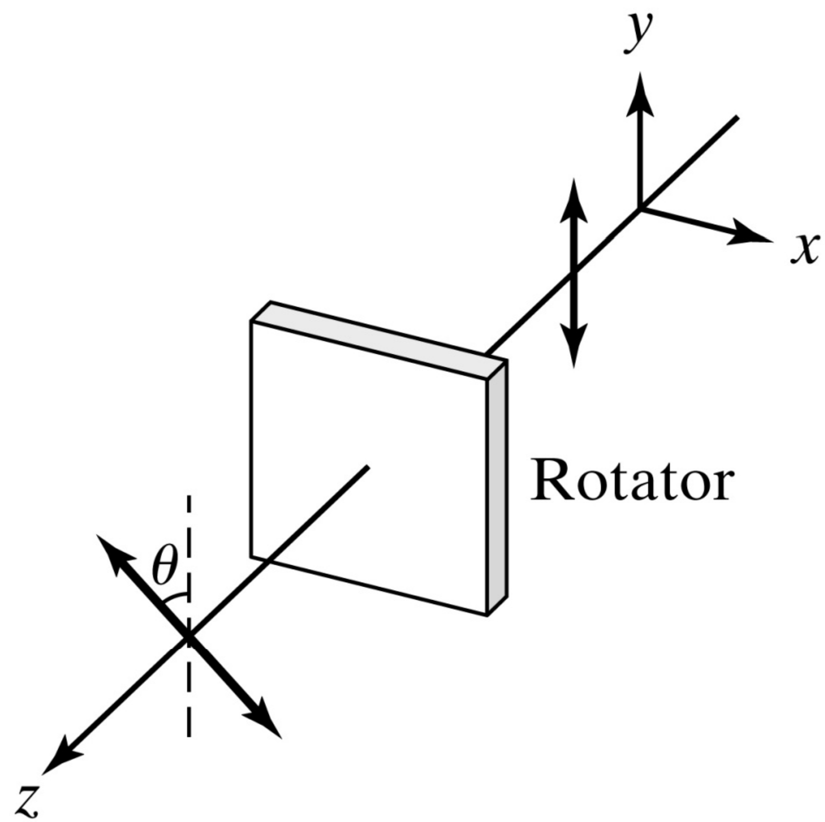


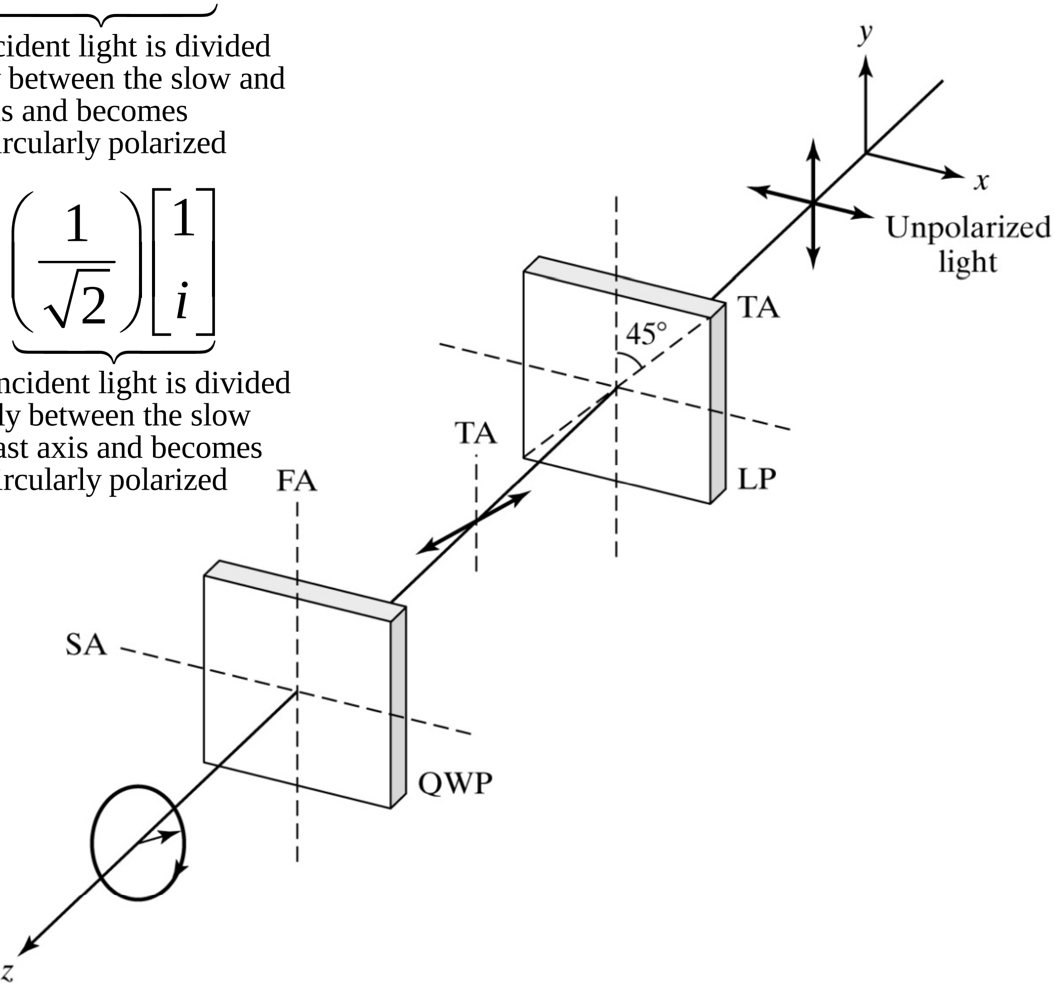
TABLE 14-2 SUMMARY OF JONES MATRICES

I. Linear polarizers					
TA horizontal	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	TA vertical	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	TA at 45° to horizontal	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
II. Phase retarders					
		General	$\begin{bmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{bmatrix}$		
QWP, SA vertical	$e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$	QWP, SA horizontal	$e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$		
HWP, SA vertical	$e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	HWP, SA horizontal	$e^{i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$		
III. Rotator					
Rotator	$(\theta \rightarrow \theta + \beta)$	$\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$			

Example: Production of circularly polarized light by combining a linear polarizer with a QWP

$$\underbrace{e^{i\pi/4}\begin{bmatrix}1 & 0\\ 0 & -i\end{bmatrix}}_{\substack{QWP\\ \text{slow axis}\\ \text{horizontal}}}\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}}_{\substack{\text{Linearly polarized}\\ \text{light at }45^0}} = e^{i\pi/4}\underbrace{\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix}1\\ -i\end{bmatrix}}_{\substack{\text{The incident light is divided}\\ \text{equally between the slow and}\\ \text{fast axis and becomes}\\ \text{right-circularly polarized}}}$$

$$\underbrace{e^{-i\pi/4}\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}}_{\substack{QWP\\ \text{slow axis}\\ \text{vertical}}}\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}}_{\substack{\text{Linearly polarized}\\ \text{light at }45^0}} = e^{-i\pi/4}\underbrace{\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix}1\\ i\end{bmatrix}}_{\substack{\text{The incident light is divided}\\ \text{equally between the slow}\\ \text{and fast axis and becomes}\\ \text{left-circularly polarized}}}$$



Example: Left-circularly polarized light is passing through an eighth wave plate

Eighth wave plate is a phase retarder that introduces a relative phase difference of $2\pi/8$ or $\pi/4$ between the SA and FA. Assume $\varepsilon_x = 0$

$$\mathbf{M} = \underbrace{\begin{bmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{bmatrix}}_{\text{Phase retarderl}} \xrightarrow[\varepsilon_y=\pi/4]{\varepsilon_x=0} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}}_{\textit{Eighth}\text{ wave plate}}$$

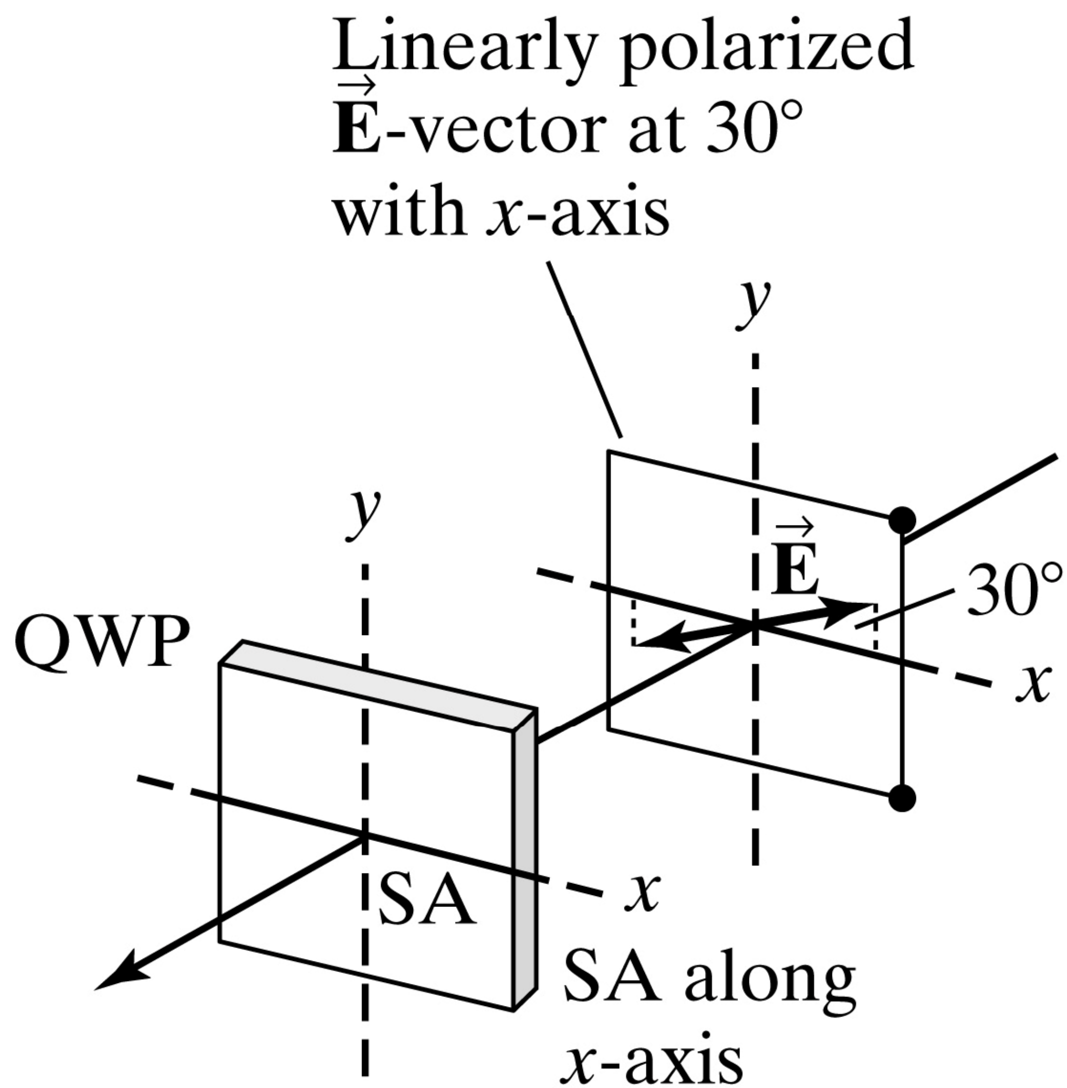
$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}}_{\textit{Eighth}\text{ wave plate}} \underbrace{\begin{bmatrix} 1 \\ i \end{bmatrix}}_{\text{Left-circularly polarized light}} = \begin{bmatrix} 1 \\ ie^{i\pi/4} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ e^{i3\pi/4} \end{bmatrix}}_{\text{Elliptically polarized light}}$$

$$e^{3\pi/4} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$\begin{bmatrix} \widetilde{E}_{0x} \\ \widetilde{E}_{0y} \end{bmatrix} = \begin{bmatrix} A \\ B + iC \end{bmatrix} \text{where } \mathrm{E}_{0x} = A = 1,\ B = -\frac{1}{\sqrt{2}} \text{ and } C = \frac{1}{\sqrt{2}},\ E_{0y} = \sqrt{B^2 + C^2} = 1 \text{ and}$$

$$\tan 2\alpha = \frac{2E_{0x}E_{oy}\cos\varepsilon}{E_{0x}^2 - E_{0y}^2} \rightarrow \alpha = -45^0$$

$A > 0$ and C>0 so they have the same sign so the elliptically polarized light has counterclockwise rotation.


Linearly polarized
$\vec{\mathbf{E}}$-vector at 30°
with x-axis
y
y
$\vec{\mathbf{E}}$
30°
QWP
x
SA
x
SA along
x-axis