Chapter 11
Fraunhofer Diffraction

Lecture Notes for Modern Optics based on
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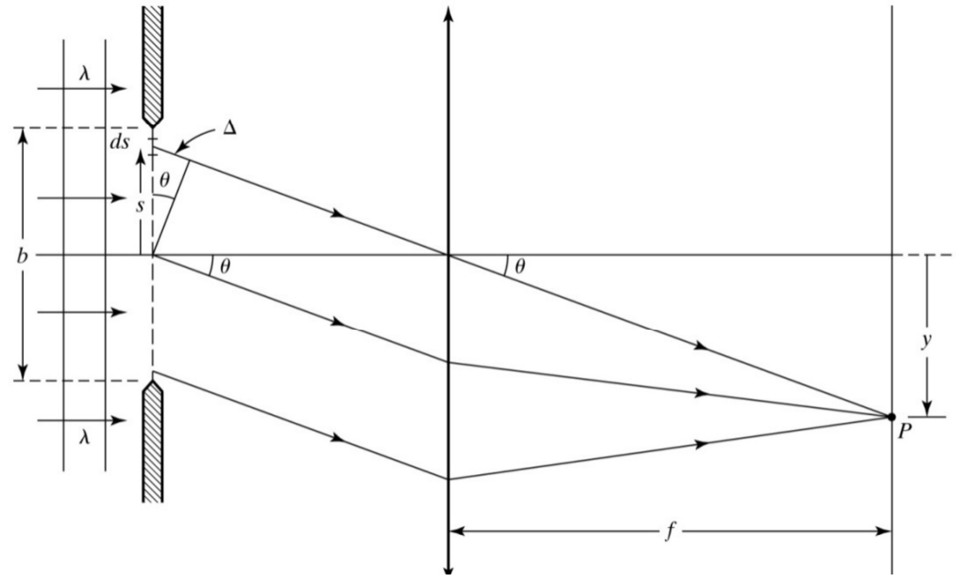
Diffraction

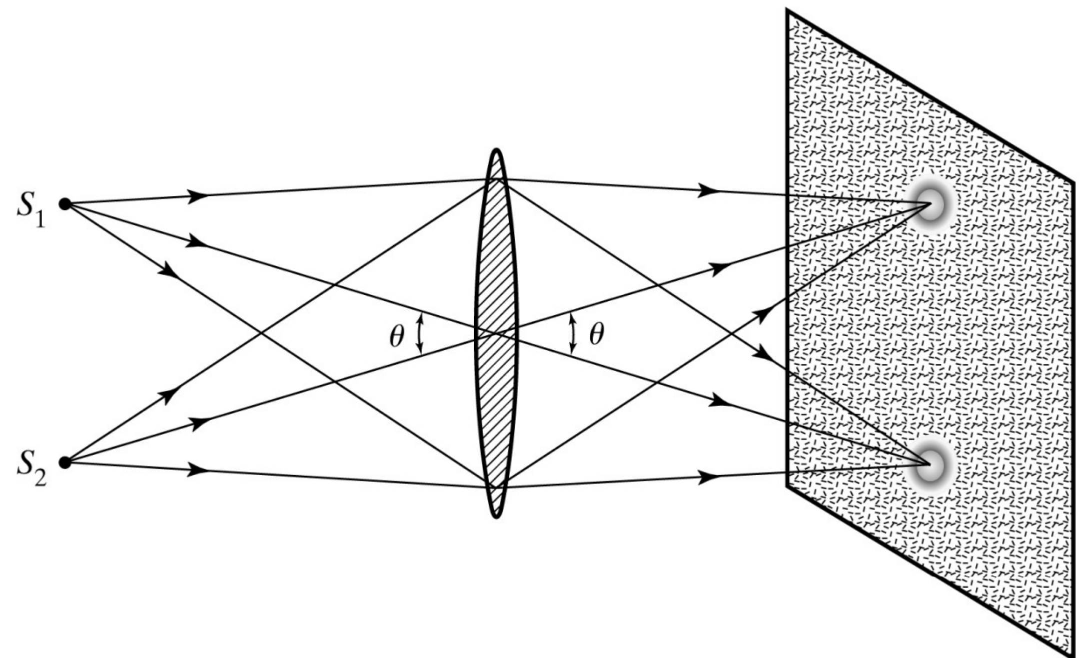
- **Diffraction** is any deviation from geometrical optics that results from obstruction of the wavefront of light.
- Obstruction causes local variations in the amplitude or phase of the wave
- Diffraction can cause image blurriness. So the aberrations.
- An optical component that is free of aberrations is called **diffraction-limited optics** and is still subject to blurriness due to diffraction.
- Huygens-Fresnel principle: every point on a wavefront can be considered as a source of secondary spherical wavelets(Huygens). The field beyond a wavefront is result of the superposition of these wavelets taking into account their amplitudes and phases (Fresnel).

Diffraction vs. interference:

Diffraction phenomena is calculated from interference of the waves originating from <u>different points of a continuous source</u>.

Interference phenomena is calculates interference of the beams origination from <u>discrete number of sources</u>.





Mathematical treatment of Diffraction

- **Far-field or Fraunhofer Diffraction or :** when both the source and observation screen are far from the diffraction causing aperture, so that the waves arriving at the aperture and screen can be **approximated by plane waves**.
- **Near-field or Fresnel diffraction:** when the curvature of the wavefronts an not be ignored
- We will only investigate the far-field diffraction in this course using the Huygens-Fresnel principle with one approximation.
- When EM waves hit the edges of the aperture there is oscillations of the electrons in the matte that cause a secondary field (**edge effect**).
- <u>In HF approximation we ignore edge effect. So beyond the aperture there is no field. This holds only if the observation point is far from the aperture.</u>

Diffraction from a single slit

We simulate the geometrical arrangement for the Frounhofer diffraction by placing a point source at the focal point of a lens and a screen at the focal point of a lens after the amerture. The light reaching point P on the screen is from interference of the parallel rays from different points on the aperture.

We consider each interval of ds as a source and calculate contribution of the all of these sources.

r: optical path length from the ds to the P

E_L strength of the electric field contribution from each point.

$r = r_0$ for the field from the center of the slit.

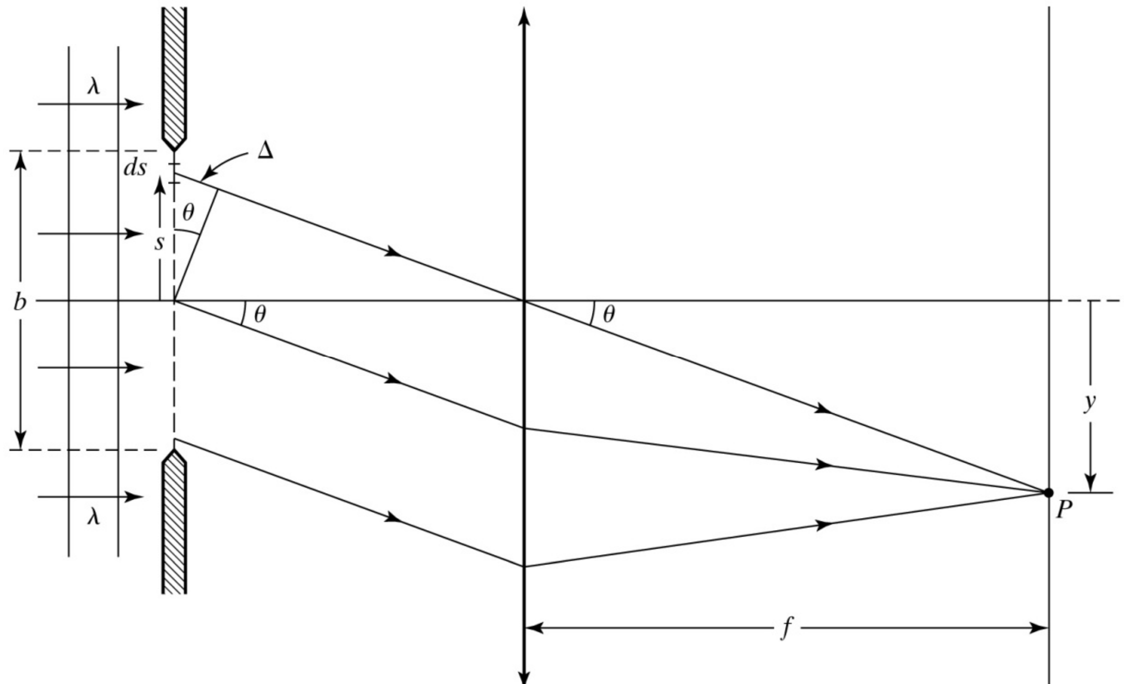
Contribution of each interval ds to the wavefront at point P is a spherical wavelet of dE_P:

$$dE_P = \underbrace{\left(\frac{E_L ds}{r}\right)}_{\text{Spherical wave amplitude}} e^{i(kr-\omega t)}$$

$$dE_P = \left(\frac{E_L ds}{r_0 + \Delta}\right) e^{i(k(r_0+\Delta)-\omega t)}$$

$$dE_P = \underbrace{\left(\frac{E_L ds}{r_0 + \Delta}\right)}_{\text{Path difference affets the amplitude}} \underbrace{e^{i(kr_0-\omega t)} e^{ik\Delta}}_{\text{Path difference affets the phase}}$$

We ignore Δ in the denuminator since $\Delta << r_0$



Diffraction from a single slit

$$dE_P = \left(\frac{E_L ds}{r_0} \right) e^{i(kr_0 - \omega t)} e^{ik\Delta}$$

The total electric field at the point P

$$E_P = \int_{slit} dE_P = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \int_{-b/2}^{\mathrm{b}/2} e^{iks\sin\theta} ds$$

$$E_P = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \left(\frac{e^{iks\sin\theta}}{ik\sin\theta} \right)_{-b/2}^{b/2} = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \frac{e^{(ikb\sin\theta)/2} - e^{-(ikb\sin\theta)/2}}{ik\sin\theta}$$

With $\beta = \frac{1}{2} kb\sin\theta$ and using Euler's formula

$$E_P = \frac{E_L b}{r_0} \frac{\sin\beta}{\beta} e^{i(kr_0 - \omega t)} = \frac{E_L b \sin c\beta}{r_0}$$

β varies with θ and that varies with the distance from the screen.

β can be interpreted as phase difference $\mathrm{k}\Delta = \mathrm{k}\frac{\mathrm{b}}{2}\sin\theta \rightarrow \boxed{|\beta| = |k\Delta| = \frac{1}{2} kb\sin\theta}$

$|\beta|$ shows the magnitude of the phase difference at point P between the points from the center and either endppoint of the slit.

The irradiance I at P is proportional to the E_0^2:

$$I = \left(\frac{\varepsilon_0 c}{2} \right) E_0^2 = \frac{\varepsilon_0 c}{2} \left(\frac{E_L b}{r_0} \right)^2 \frac{\sin^2\beta}{\beta^2} \text{ with } \boxed{\mathrm{I}_0 = \frac{\varepsilon_0 c}{2} \left(\frac{E_L b}{r_0} \right)^2} \text{ we have } \boxed{I = I_0 \frac{\sin^2\beta}{\beta^2} = I_0 \sin c^2 \beta}$$

Diffraction from a single slit

The diffraction pattern of the light from a silt by width of b at a distanc screen $I = I_0 \dfrac{\sin^2 \beta}{\beta^2} = I_0 \sin c^2 \beta$ where

$|\beta| = |k\Delta| = (kb \sin\theta)/2$ is the phase difference betwwen the light from the center of the slit the endpoints.

Let's plot the I. <u>For the central maximum</u>: $\lim_{\beta \to 0} \sin c(\beta) = \lim_{\beta \to 0}\left(\dfrac{\sin\beta}{\beta}\right) = 1$

<u>For zeros of the sinc function</u>:

$\sin\beta = 0 \rightarrow \beta = \dfrac{1}{2}(kb\sin\theta) = m\pi \ \rightarrow b\sin\theta = m\lambda$ where $m = \pm 1, \pm 2, \ldots$

$m = 0$ does not lead to a zero be cause of the special property of the sinc function.

if f is the distance of the slit from the screen, then the location of the minima on the screen can be found using small-angle approximation:

$\sin\theta \cong \tan\theta = y/f \ \rightarrow \dfrac{\mathrm{m}\lambda}{\mathrm{b}} = \dfrac{y_m}{f} \rightarrow \boxed{y_m = \dfrac{m\lambda f}{b}}$

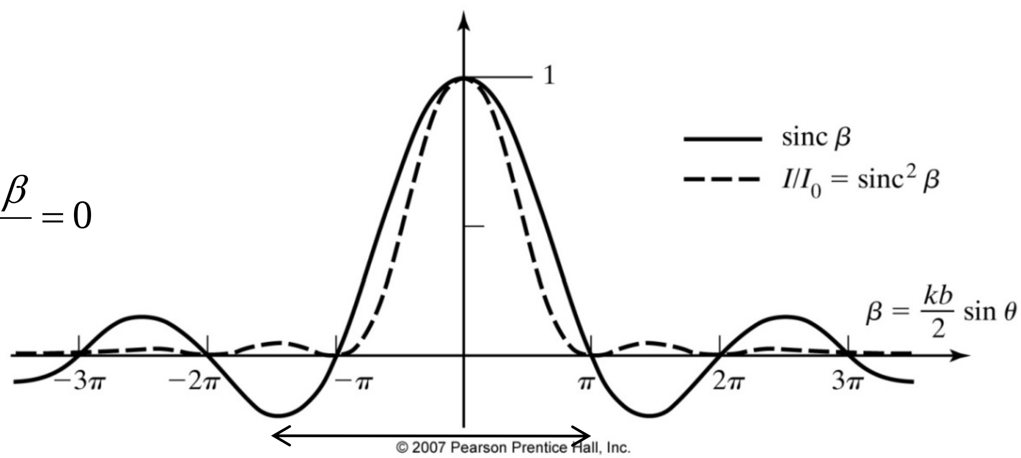
<u>For the other maxima</u>: $\dfrac{\mathrm{d}}{\mathrm{d}\beta}\left(\dfrac{\sin\beta}{\beta}\right) = \dfrac{\beta\cos\beta - \sin\beta}{\beta^2} = 0$

<u>Cental lobe</u>: image of the slit with roundes edges.

<u>Side lobes</u>: what causes blurriness in the image.

<u>Angular width of the central lobe</u>: $b\sin\theta = \lambda \rightarrow$

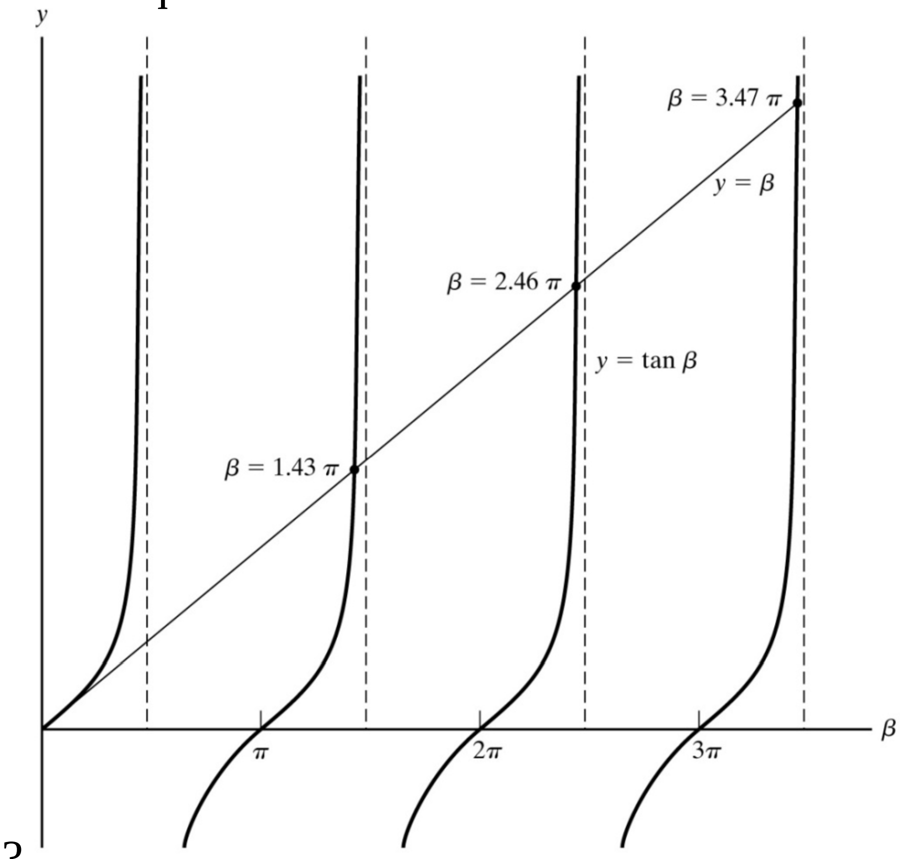
$\theta_1 \cong \dfrac{\lambda}{b},\ \theta_{-1} \cong \dfrac{\lambda}{b}, \rightarrow \boxed{\Delta\theta = \dfrac{2\lambda}{b}}$ The central lobe will sprad as the slit-size gets smaller.



Maxima of the sinc function

The maxima of the sinc function are solutions of this equation:

$$\frac{\beta\cos\beta-\sin\beta}{\beta^2}=0\rightarrow\tan\beta=\beta$$



We can solve this equation by parametrically.

An angle equal to its tangent is intersection

of the $\begin{cases} y=\beta \\ y=\tan\beta \end{cases}$

We see that the maxima are not exactly half way between the minima. They occur slightly erlier and as β increases the maxima shift towards the center.

Ratio of the irradiances at the central peak maximum to the first of the secondary maxima?

$$\frac{I_{\beta=0}}{I_{\beta=1.43\pi}}=\frac{\left(\sin\beta/\beta\right)^2_{\beta=0}}{\left(\sin\beta/\beta\right)^2_{\beta=1.43\pi}}=\frac{1}{\left(\sin\beta/\beta\right)^2_{\beta=1.43\pi}}=\left(\frac{1.43\pi}{\sin\left(1.43\pi\right)}\right)^2=\frac{20.18}{0.952}$$

$\boxed{I_{\beta=1.43\pi}=0.047I_{\beta=0}}$ or only 4.7% of the I_0

Beam spreading

The angular spread of the central maximm $\Delta\theta = 2\lambda/b$ is independent of the distance between the slit and screen. So as screen moves away from the slit the nature of the diffraction pattern does not change.

W is the width of the central maximum,

$$\boxed{W = L\Delta\theta = \frac{2L\lambda}{b}}$$

All of the beams spread according to diffraction as they propagate due to the finite size of the source.

Even if we make them parallel with a lens still they will spread because of the diffraction.

Example:

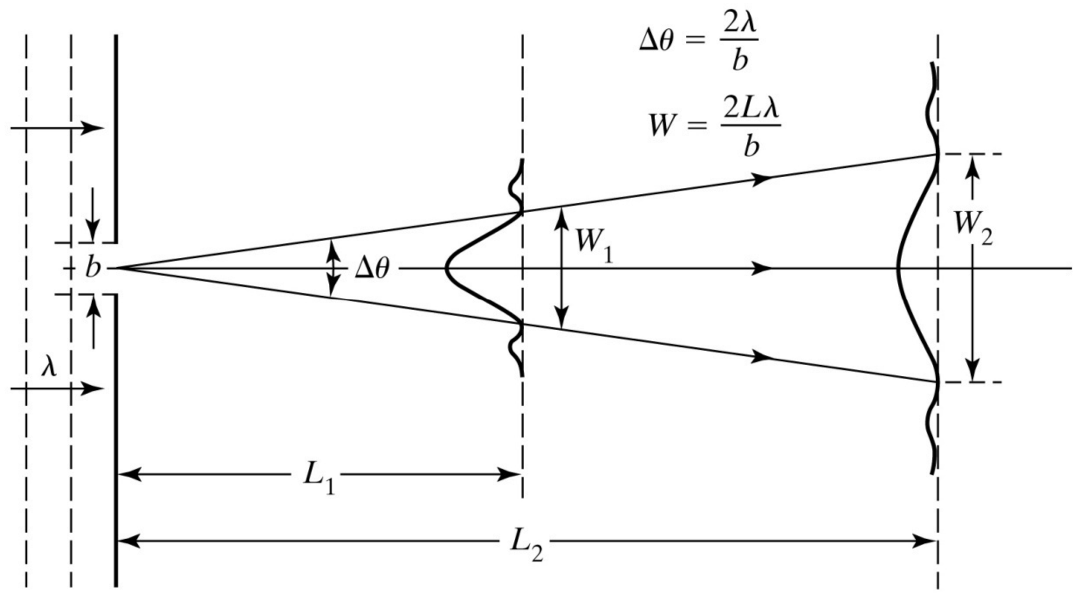
What is the width of a parallel beam of $\lambda = 546nm$ and width of $\boxed{b=0.5mm}$ after propagation of 10 meter?

$$W = \frac{2L\lambda}{b} = \frac{2\times10\times546\times10^{-9}}{0.5\times10^{-3}}$$

$$\boxed{W = 21.8mm}$$

This treatment of beam spreading is correct for far-field where $L >> \dfrac{b^2}{\lambda}$

or more generally $L >> \dfrac{\text{area of aperture}}{\lambda}$



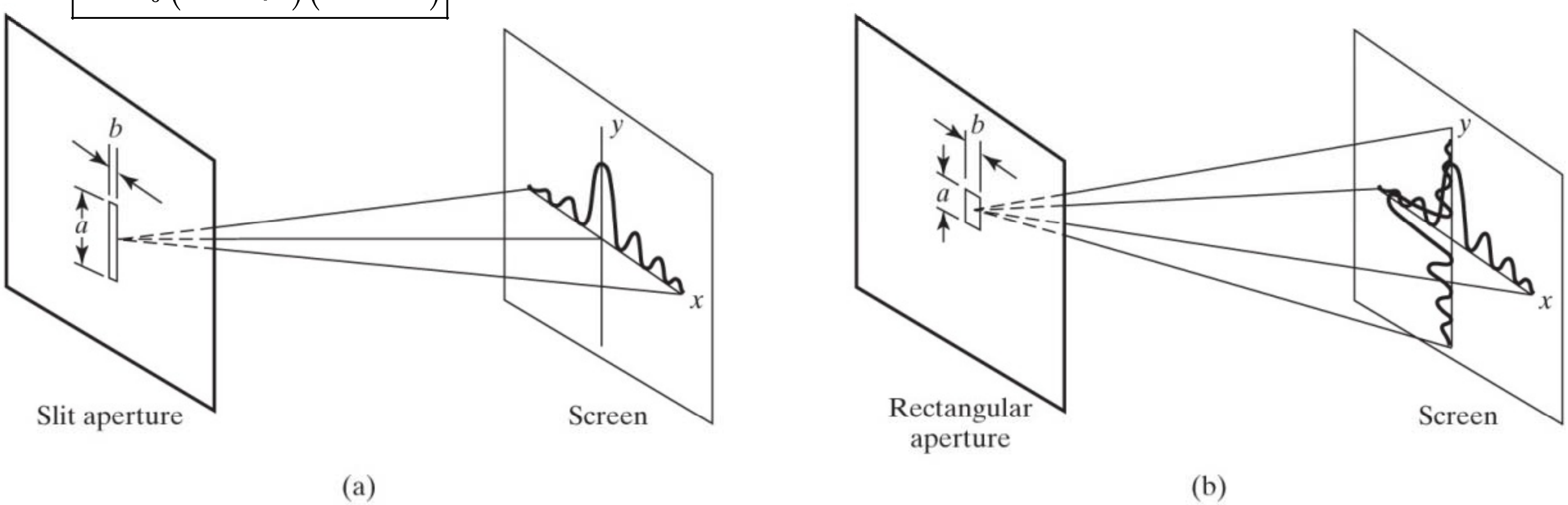
Rectangular aperture

For a slit of length a and width b we calculate the diffraction pattern. We assumed a>>b in previous section. When a and b are comparable and both small we have large contriibutions to the diffraction pattern from both dimensions: $I = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2$ *where* $\alpha = \left(\frac{k}{2}\right)a\sin\theta$ and $I = I_0\left(\frac{\sin\beta}{\beta}\right)^2$ *where* $\beta = \left(\frac{k}{2}\right)b\sin\theta$

Zeros of the pattern due occur at: $\boxed{\mathrm{y_m} = \frac{m\lambda f}{b} \text{ and } \mathrm{x_n} = \frac{n\lambda f}{a} \quad \text{m,n}=\pm 1, \pm 2, \ldots}$

Using a bit more analysis we can write $E_P = \int_{slit} dE_P = \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \int_{-a/2}^{a/2} e^{ikXx/r_0} dx \int_{-b/2}^{\mathrm{b}/2} e^{ikYy/r_0} dy$

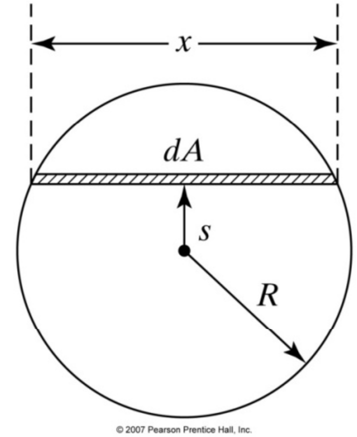
X, Y are the coordinates of the observation point on the screen and x,y are the coordinates of the surface element on the aperture. The total irradiance turns out to be the product of the irradiance functions on each dimension: $\boxed{I = I_0\left(\sin c^2\beta\right)\left(\sin c^2\alpha\right)}$



Circular aperture

For finding the diffraction pattern caused by a circular aperture we assume the incremental electric field amplitude at point P due to the surface element $dA = dxdy$ (on the aperture) is $E_A dA / r_0$. Then we integrate the incremental field over the entire aperture area. The resulting E at point P is

$$dE_P = \left(\frac{E_A dA}{r_0 + \Delta}\right) e^{i(k(r_0+\Delta)-\omega t)} \xrightarrow[\Delta = s\sin\theta]{\Delta << r_0} E_P = \frac{E_A}{r_0} e^{i(kr_0 - \omega t)} \iint_{Area} e^{isk\sin\theta} dA$$



By choosing the elemental area the rectangular area of $dA = xds$ we can reduce the double integral to a single one.

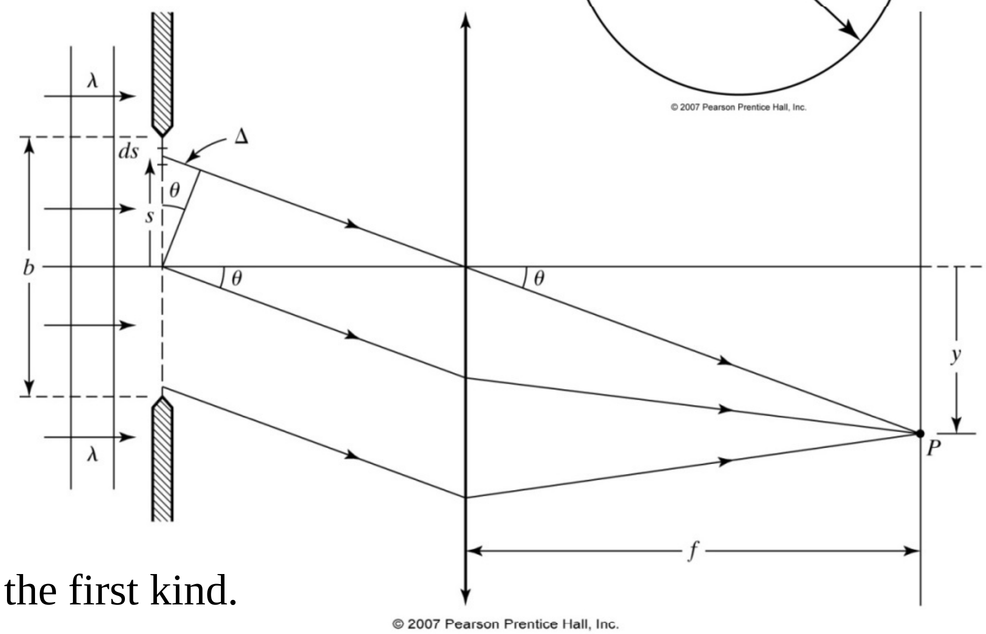
$$\left(\frac{x}{2}\right)^2 + s^2 = R^2 \rightarrow x = 2\sqrt{R^2 - s^2}$$

$$E_P = \frac{2E_A}{r_0} e^{i(kr_0 - \omega t)} \int_{-R}^{+R} e^{isk\sin\theta\sqrt{R^2 - s^2}} ds$$

Substituting $v = s/R$ and $\gamma = k\sin\theta$

$$E_P = \frac{2E_A R}{r_0} e^{i(kr_0 - \omega t)} \int_{-1}^{+1} e^{i\gamma v\sqrt{1-v^2}} dv$$

From the intergral table: $\int_{-1}^{+1} e^{i\gamma v\sqrt{1-v^2}} dv = \frac{\pi J_1(\gamma)}{\gamma}$



where $J_1(\gamma)$ is the first order Bessel function of the first kind.

$$J_1(\gamma) = \frac{\gamma}{2} - \frac{(\gamma/2)^3}{1^2.2} + \frac{(\gamma/2)^5}{1^2.2^2.3} - \cdots \text{ and } \lim_{\gamma\to 0} \frac{J_1(\gamma)}{\gamma} = \frac{1}{2}$$

Field from an arbitrary shape aperture on a distant screen

For finding the diffraction pattern caused by an aperture of arbitrary shape we assume the incremental electric field amplitude at point P due to the surface element $dA = dxdy$ (on the aperture) is $E_A dA / r_0$. Then we integrate the incremental field over the entire aperture area. The resulting E at point P is

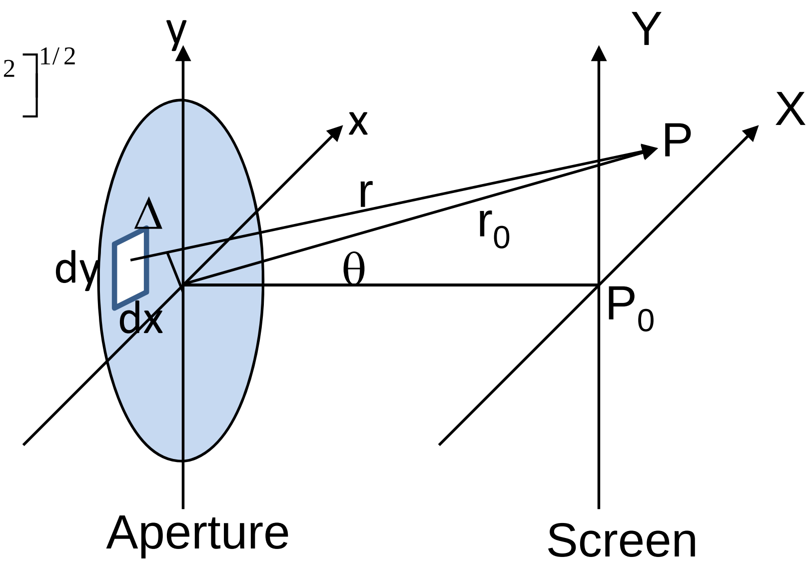
$$E_p = \frac{E_A}{r_0} \iint\limits_{Aperture} e^{i(\omega t - kr)} dA \quad \text{where } dA = dxdy$$

$$r = \left[(X-x)^2 + (Y-y)^2 + Z^2 \right]^{1/2} \quad \text{with } r_0 = \left[X^2 + Y^2 + Z^2 \right]^{1/2}$$

we have: $r = r_0 \left[1 + \frac{(x^2+y^2)}{r_0^2} - \frac{2(Xx+Yy)}{r_0^2} \right]^{1/2}$

$$r_0^2 >> x^2 + y^2 \rightarrow r \simeq r_0 \left[1 - \frac{2(Xx+Yy)}{r_0^2} \right]^{1/2}$$

$$r_0 >> x + y \rightarrow \boxed{r \simeq r_0 - \frac{(Xx+Yy)}{r_0}} \text{ only first two terms.}$$



$$E_P = \frac{E_A}{r_0} \iint\limits_{Aperture} e^{i\left(\omega t - k\left(r_0 - \frac{(Xx+Yy)}{r_0}\right)\right)} dA \rightarrow E_P = \frac{E_L}{r_0} e^{i(\omega t - kr_0)} \iint\limits_{Aperture \text{ any shape}} e^{ik(Xx+Yy)/r_0} dxdy$$

For specific shape of aperture the integral limits and relationship between x and y will change. We will look at two examples of rectangular and circular apertures.

Rectangular aperture

$$E_P = \frac{E_A}{r_0} e^{i(\omega t - kr_0)} \iint\limits_{\substack{Aperture \\ \text{any shape}}} e^{ik(Xx+Yy)/r_0}\, dxdy$$

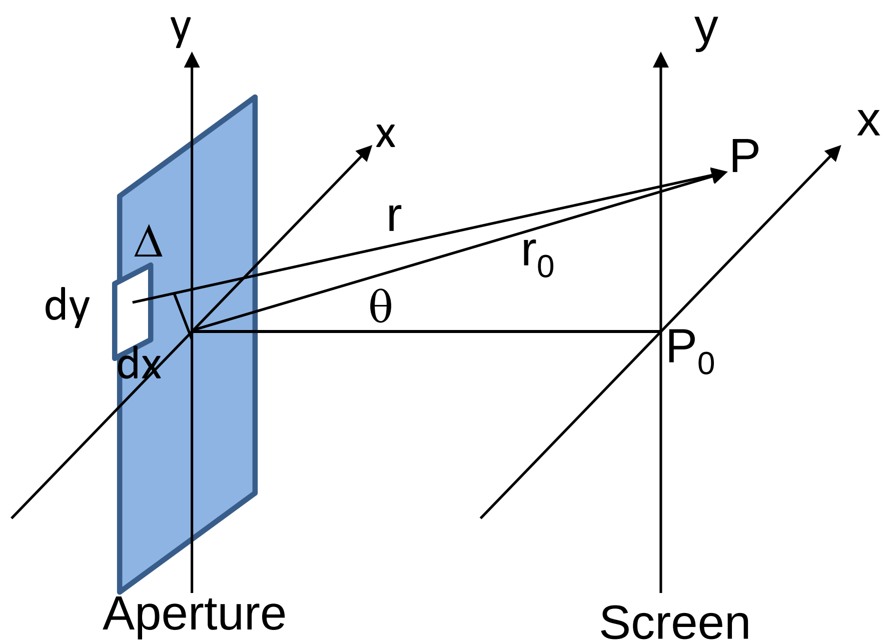
$$E_P = \frac{E_A}{r_0} e^{i(\omega t - kr_0)} \int_{-b/2}^{b/2} e^{ik(Yy)/r_0} dy \int_{-a/2}^{a/2} e^{ik(Xx)/r_0} dx$$

$$E_p = \frac{E_A}{r_0} e^{i(\omega t - kr_0)} b\left(\frac{\sin\beta}{\beta}\right) a\left(\frac{\sin\alpha}{\alpha}\right)$$

$$E_p = \frac{AE_A}{r_0} e^{i(\omega t - kr_0)} \left(\frac{\sin\beta}{\beta}\right)\left(\frac{\sin\alpha}{\alpha}\right)$$

$$I_0 = \left(\frac{AE_A}{r_0} e^{i(\omega t - kr_0)}\right)^2$$

$$\boxed{I(X,Y) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin\alpha}{\alpha}\right)^2}$$



Where $\boxed{\alpha = \frac{k}{2} a \frac{X}{r_0} = \frac{\mathrm{k}}{2} \mathrm{a}\sin\theta_x}$ and $\boxed{\beta = \frac{k}{2} b \frac{Y}{r_0} = \frac{\mathrm{k}}{2} \mathrm{b}\sin\theta_y}$

Circular aperture I

For circular aperture we introduce the spherical coordinates because of the symmetry of the problem. On plane of the aperture: $x = \rho\cos\phi,\ y = \rho\sin\phi$

On plane of the screen: $\mathrm{X} = q\cos\Phi,\ \mathrm{Y} = q\sin\Phi$

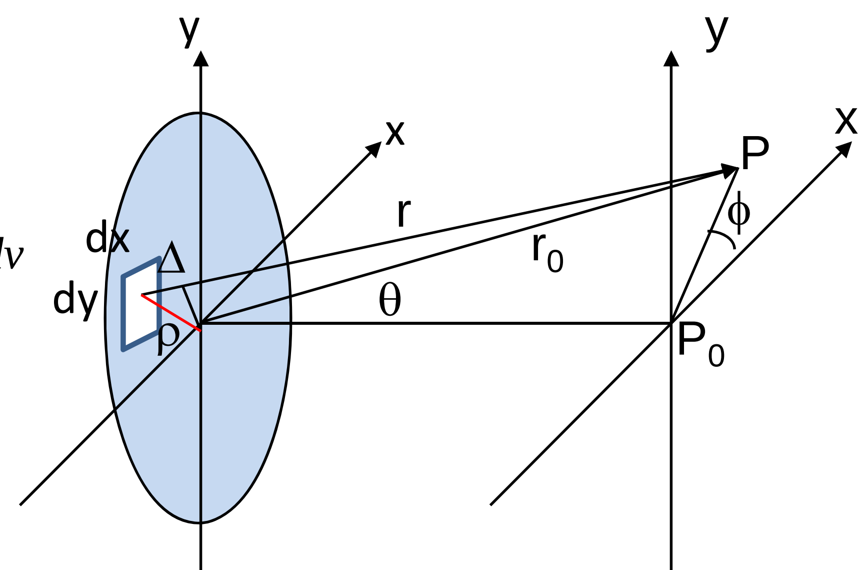
Differential surface element: $dA = \rho d\rho d\phi$

$$E_P = \frac{E_A}{r_0} e^{i(\omega t - kr_0)} \underset{\substack{Aperture\\ \text{any shape}}}{\iint} e^{ik(Xx+Yy)/r_0} dA = \frac{E_L}{r_0} e^{i(\omega t - kr_0)} \int_{\rho=0}^{R}\int_{\phi=0}^{2\pi} e^{i(k\rho q/r_0)\cos(\phi-\Phi)} \rho d\rho d\phi$$

Wher we used: $Yy + Xx = \rho q(\sin\phi\sin\Phi + \cos\phi\cos\Phi) = \rho q\cos(\phi - \Phi)$

Because of the axial symmetry of the problem, the solution must be independent of Φ and since point P is an arbitrary point on the screen we choose to have $\Phi = 0$ to simplify things.

$$\mathrm{E_P} = \frac{E_A}{r_0} e^{i(\omega t - kr_0)} \int_{\rho=0}^{a} \rho d\rho \underbrace{\int_{\phi=0}^{2\pi} e^{i(k\rho q/r_0)\cos(\phi)} d\phi}_{\text{A tabulated integral. Solution is a Bessel function of the first kind}}$$



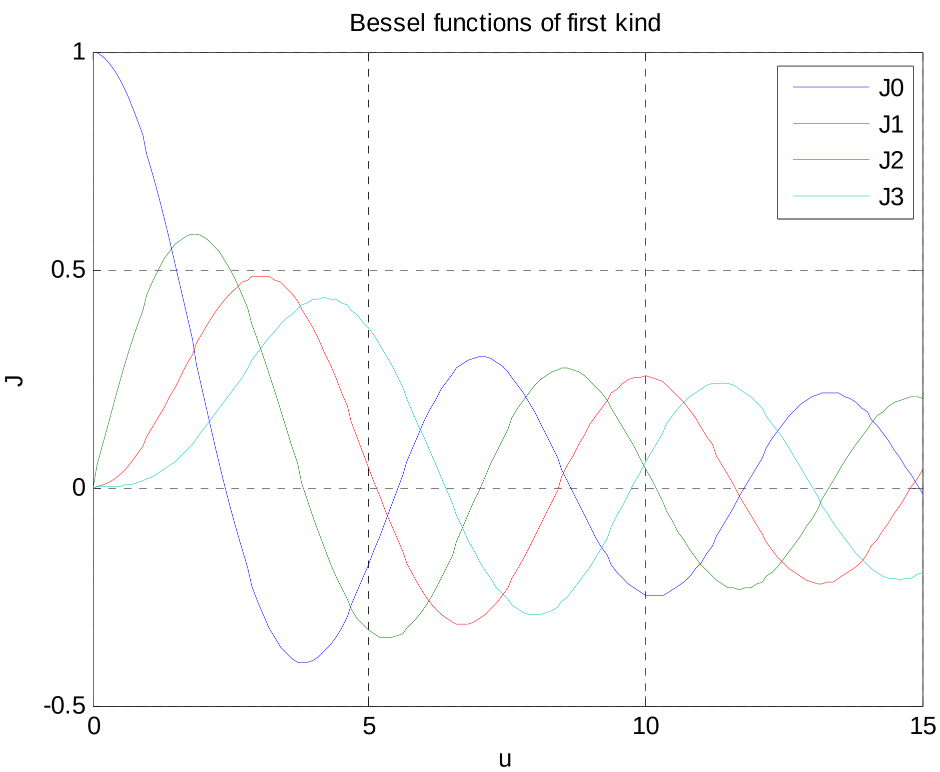
Bessel function 1st kind: $J_m(u) = \frac{i^{-m}}{2\pi}\int_0^{2\pi} e^{i(mv + u\cos v)} dv$

For m=0: $J_0(u) = \frac{1}{2\pi}\int_0^{2\pi} e^{iu\cos v} dv$ and with $u = k\rho q$

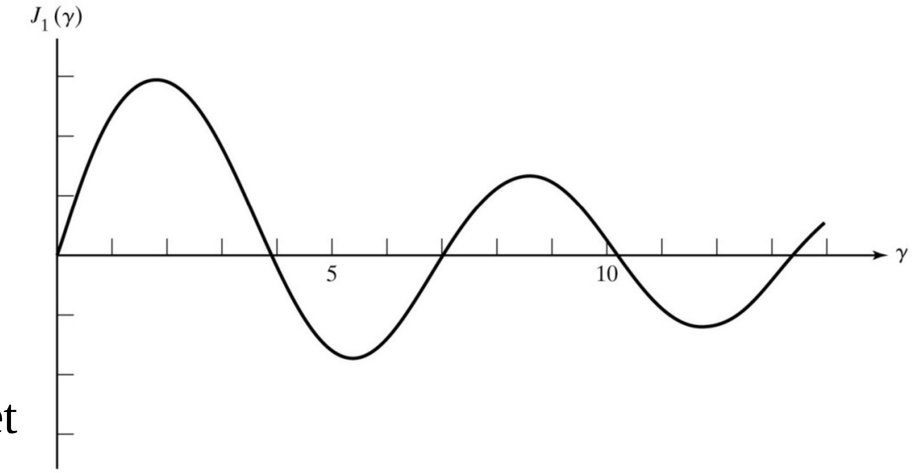
$$\mathrm{E_P} = \frac{E_A}{r_0} e^{i(\omega t - kr_0)} \int_{\rho=0}^{R} J_0(k\rho q / r_0)\rho d\rho$$

Bessel functions of the first kind (MATLAB)

```
u = (0:0.1:15)
BJ0=besselj(0,u);
BJ1=besselj(1,u);
BJ2=besselj(2,u);
BJ3=besselj(3,u)
plot(u,BJ0,u,BJ1,u,BJ2,u,BJ3);
legend('J0','J1','J2','J3')
title('Bessel functions of first kind');
xlabel('u'),ylabel('J')
grid on
```



Circular aperture II



$$\mathrm{E_P} = \frac{E_A}{r_0} e^{i(\omega t - kr_0)} 2\pi \int_{\rho=0}^{R} J_0\left(k\rho q / r_0\right)\rho d\rho$$

Using the one of many properties of the

bessel functions: $\int_0^u u' J_0\left(u'\right) du' = uJ_1 u$ we get

$$\int_{\rho=0}^{R} J_0\left(k\rho q / r_0\right)\rho d\rho = \left(r_0 / kq\right)^2 \int_{u'=0}^{\mathrm{kRq/r_0}} J_0\left(u'\right) u' du' = \left(kRq / r_0\right) J_1\left(kRq / r_0\right)$$

$$\mathrm{E_P} = \frac{E_A}{r_0} e^{i(\omega t - kr_0)} 2\pi R^2 \frac{r_0}{kRq} J_1\left(\frac{kRq}{r_0}\right) \rightarrow \boxed{\mathrm{E_P} = \frac{2AE_A}{r_0} e^{i(\omega t - kr_0)} \frac{r_0}{kRq} J_1\left(\frac{kRq}{r_0}\right)}$$

we use $\boxed{\gamma = kRq / r_0 = kR\sin\theta}$ and calculate the irradiance at:

$$I = \left\langle \left(\mathrm{Re}(E_P)\right)^2 \right\rangle = \frac{1}{2} E_P E_P^* \rightarrow \boxed{I = \frac{2E_A^2 A^2}{{r_0}^2}\left[\frac{J_1\left(\gamma\right)}{\gamma}\right]^2}$$

Series expansion of the $J_1\left(\gamma\right) = \frac{\gamma}{2} - \frac{\left(\gamma / 2\right)^3}{1^2.2} + \frac{\left(\gamma / 2\right)^5}{1^2.2^2.3} - \cdots \rightarrow \quad \lim_{\gamma \to 0} \frac{J_1\left(\gamma\right)}{\gamma} = \frac{1}{2}$

J_1 behaves like a damping sin function and $\mathrm{J}_1\left(\gamma\right)/\gamma$ behaves like the sinc function.

Diffraction pattern of the circular aperture with Bessel functions in MATLAB

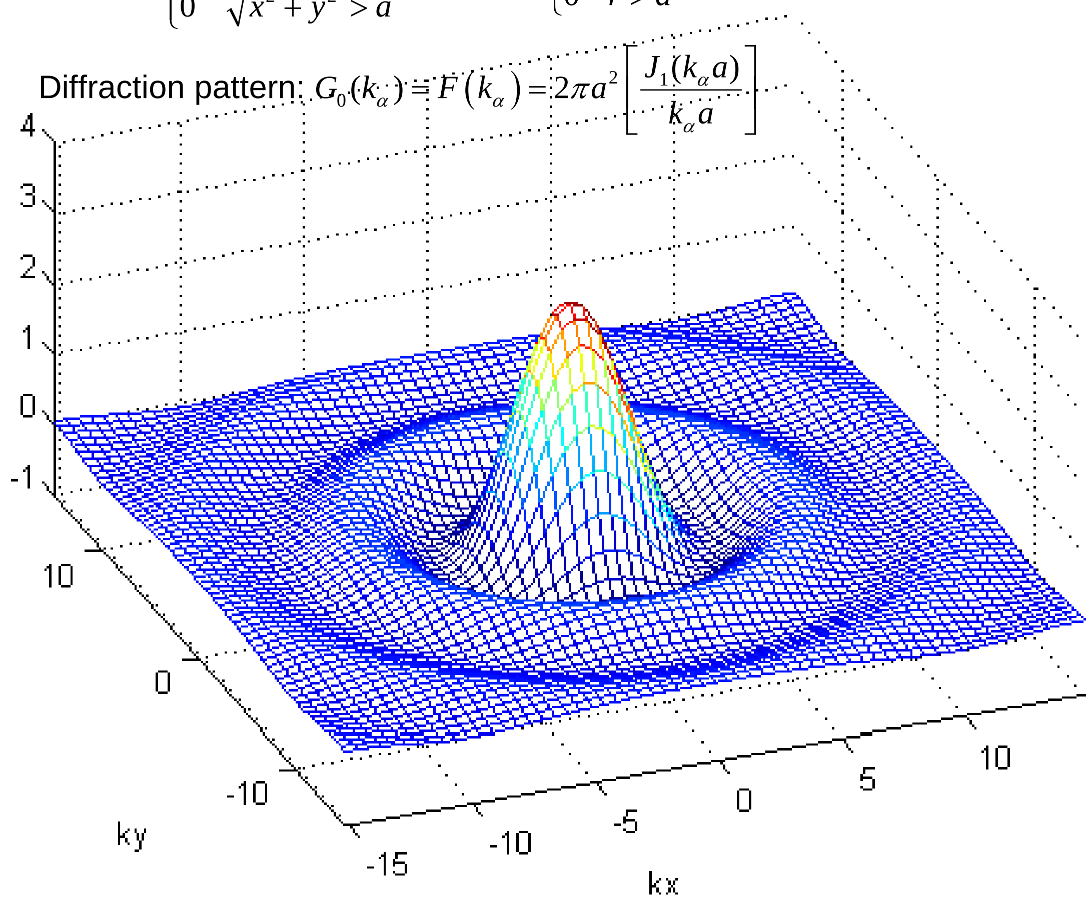
```
x=(-15.1:0.5:14.9);
y=(-15.1:0.5:14.9);
A=y.'*x;
i_index=0;
for i=-15.1:0.5:14.9
   j_index=0;
   i_index=i_index+1;
   for j=-15.1:0.5:14.9
      j_index=j_index+1;
      r=sqrt(i^2+j^2);
      if r <=5
         A(i_index,j_index)=1;
      else A(i_index,j_index)=0;
      end
   end
end
subplot(2,1,1);
mesh(x,y,A);
title('Circular Aperture')
axis([-15.1 14.9 -15.1 14.9 0 1]);
a=1;
kx=(-15.1:0.5:14.9);
ky=(-15.1:0.5:14.9);
[kax,kay]=meshgrid(kx,ky);

ka=sqrt(kax.^2+kay.^2);
Gka=2*pi*a^2.*besselj(1,ka)./(ka*a);
subplot(2,1,2);
mesh(kx,ky,Gka);
xlabel('kx'); ylabel('ky');
axis([-15.1 14.9 -15.1 14.9 -1 4]);
title('Fourier Bessel of Circular Aperture')
```

Fraunhofer diffraction pattern is the Fourier transform of the aperture function.

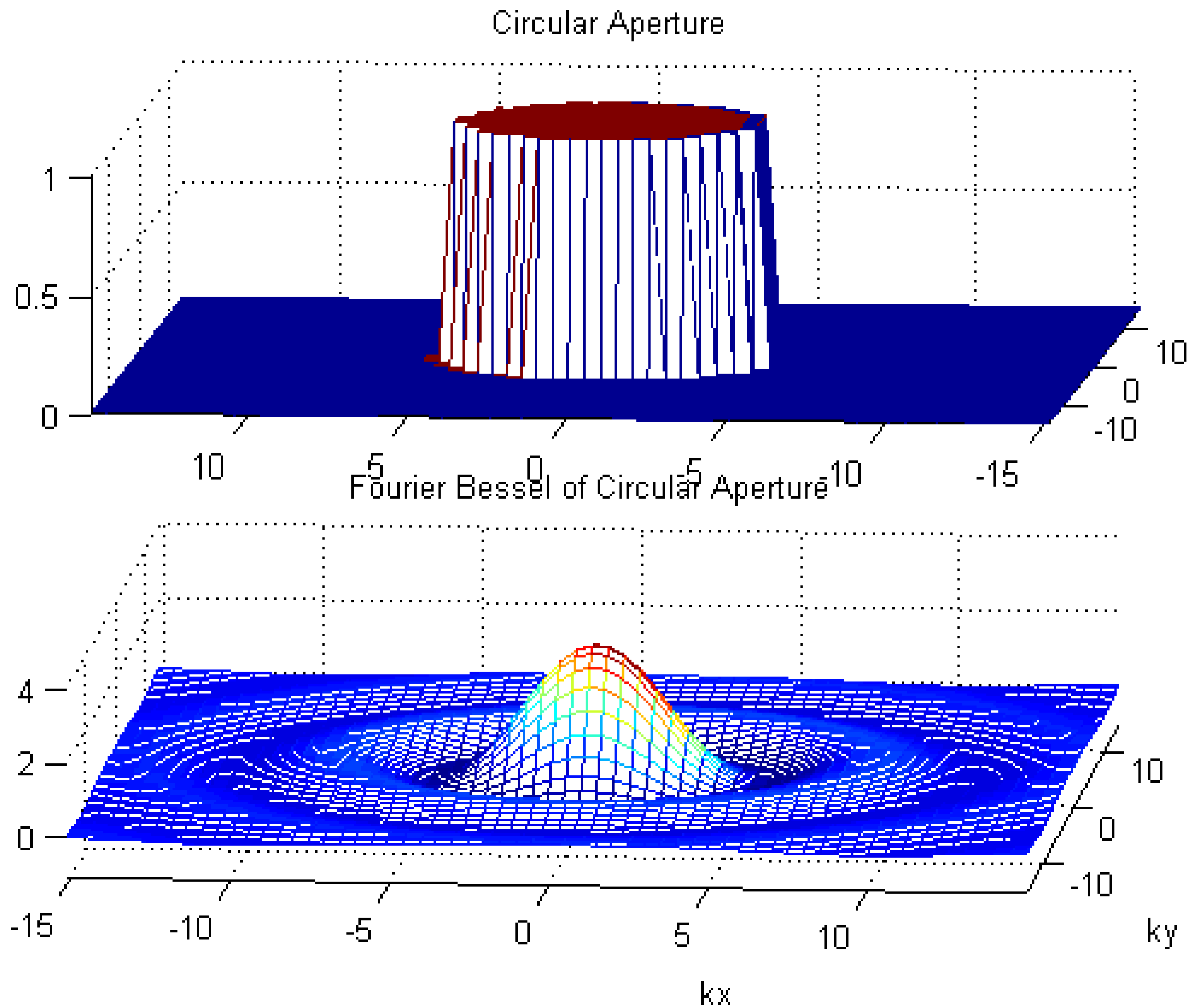
We can show this by the following plots using MATLAB. This topic will be covered in PHYS 258.

$$g(x,y)=\begin{cases}1 & \sqrt{x^2+y^2}\le a\\ 0 & \sqrt{x^2+y^2}>a\end{cases}\rightarrow g_R(r)=\begin{cases}1 & r\le a\\ 0 & r>a\end{cases}$$

Diffraction pattern: $G_0(k_\alpha)=F(k_\alpha)=2\pi a^2\left[\frac{J_1(k_\alpha a)}{k_\alpha a}\right]$

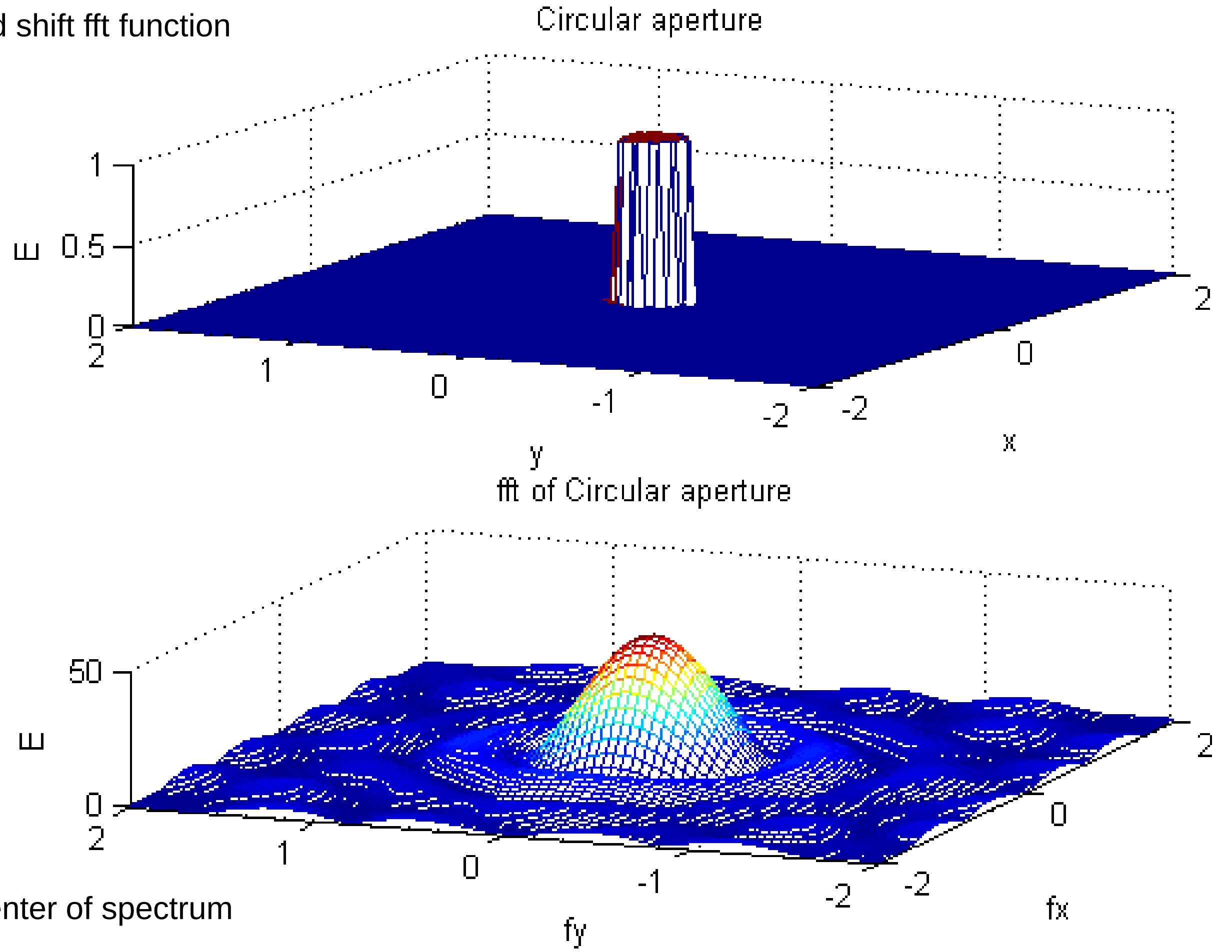


Diffraction pattern of the circular aperture with Bessel functions in MATLAB



Diffraction pattern of the circular aperture with Fast Fourier Transformation(FFT) in MATLAB

```
%PHYS 258 spring 07, Nayer Eradat
%A program to plot a circular aperture function
%and its Fourier transform using fft and shift fft function
x=(-2:0.05:2);
y=(-2:0.05:2);
A=y.'*x;
i_index=0;
for i=-2:0.05:2
   j_index=0;
   i_index=i_index+1;
   for j=-2:0.05:2
      j_index=j_index+1;
      r=sqrt(i^2+j^2);
      if r <=0.2
         A(i_index,j_index)=1;
      else A(i_index,j_index)=0;
      end
   end
end
subplot(2,1,1);
mesh(x,y,A); %3D plot
xlabel('x'); ylabel('y'); zlabel('E');
title('Circular aperture');
fft_v=abs(fft2(A));
fft_val=fftshift(fft_v);
%shift zero-frequency component to center of spectrum
subplot(2,1,2);
mesh(x,y,fft_val);
xlabel('fx'); ylabel('fy'); zlabel('E');
title('fft of Circular aperture');
```



Circular aperture diffraction pattern

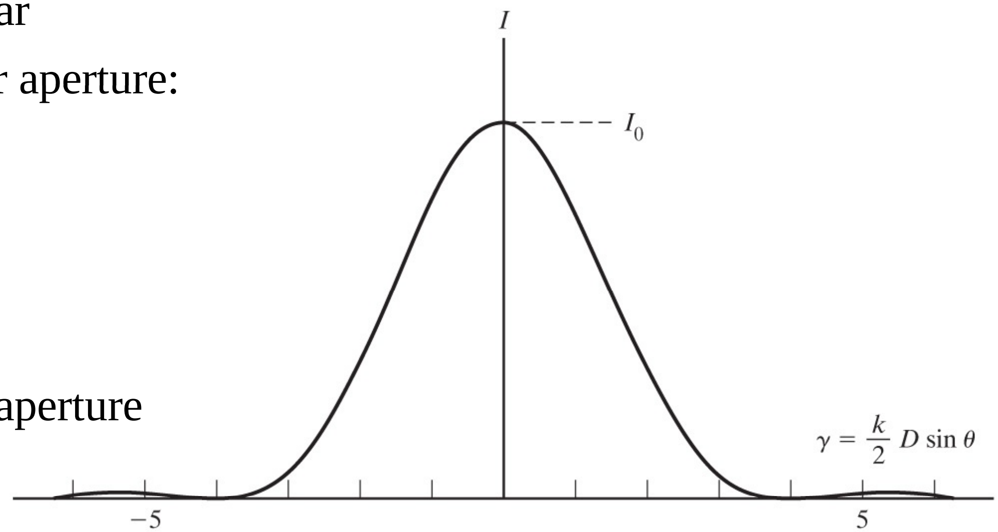
Circular aperture better than slit or rectangular aperture for imaging.First zero of the circular aperture: γ=3.832

Thus for an aperture of diameter D we have

$$\gamma=\frac{1}{2}kD\sin\theta = 3.832 \to D\sin\theta = 1/22\lambda$$

Airy Disc: the diffracted image of a circular aperture or the central lobe of the diffraction pattern.

For far-field $\sin\theta \simeq \theta$ and the angular half-width of the Airy disc is: $\boxed{\Delta\theta_{1/2} = \frac{1.22\lambda}{D}}$



(a)

	γ	$I/I_0 = (2J_1(\gamma)/\gamma)^2$
1st Maximum	0	1
1st Zero	3.832	0
2nd Maximum	5.136	0.0175
2nd Zero	7.016	0
3rd Maximum	8.417	0.00416
3rd Zero	10.173	0
4th Maximum	11.620	0.00160
4th Zero	13.324	0

(b)

Example:

The beam spread for a beam (λ=564nm) from an amperure D=0.5 mm at L=10m.

The diameter of the Airy Disc is:

$\Delta\theta_{1/2} = 1.22\lambda / D = 1.33\times 10^{-3} rad$ and then

$r_d = L\Delta\theta_{1/2} = 13mm$

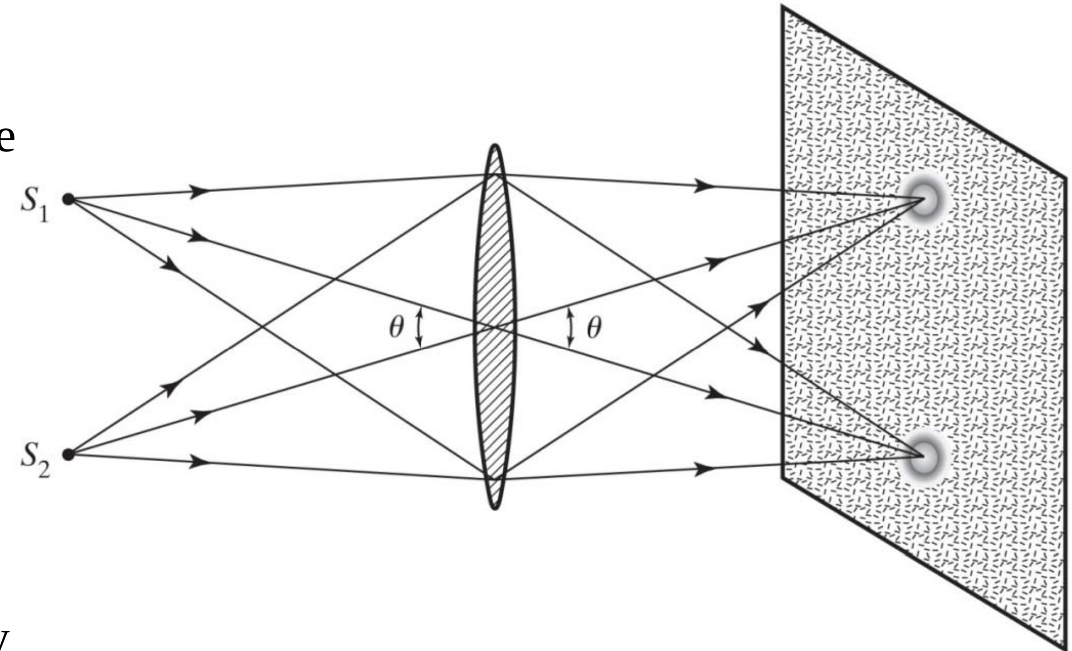
That is why we pay high $$ for the large lenses. They ca provide better image resolution.

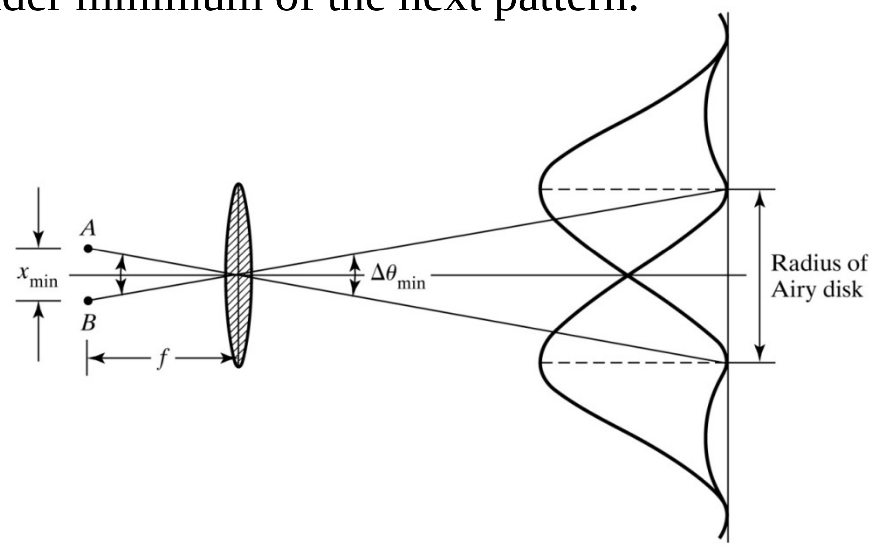
Resolution and Rayleigh criterion

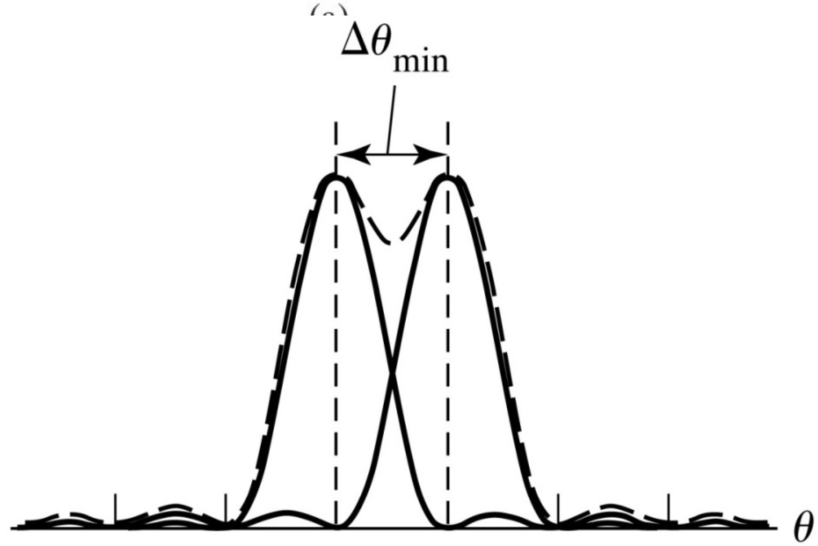
Rayleigh's criterion for just-resolvable images requires that the angular separation of the centers of the image pattern be no less than the angular radius of the airy disc.

$$\Delta\theta_{\min} = \frac{1.22\lambda}{D}$$

Maximum of one pattern falls directly under minimum of the next pattern.

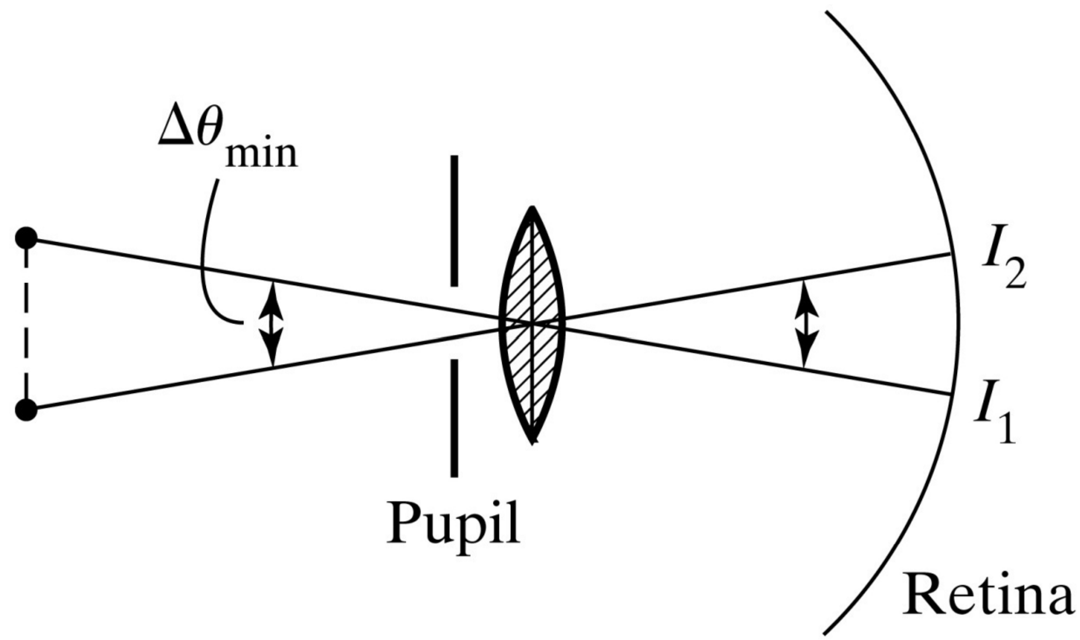






Diffraction by the eye

Pupil size limits the resolution of image of the objects subtended by $\Delta\theta_{min}$



Double-slit diffraction I

Diffraction pattern of a plane wavefront that is obstructed everywhere but at the two slits shown in fig.

We follow our analysis for the single slit. The diffraction pattern for two slits is going to be superposition of the patterns by each slit. Therefore we write:

$$E_P = \int_{slit1} dE_{P1} + \int_{slit2} dE_{P2} = \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \int_{-(1/2)(a+b)}^{-(1/2)(a-b)} e^{iks\sin\theta} ds + \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \int_{(1/2)(a-b)}^{(1/2)(a+b)} e^{iks\sin\theta} ds$$

$$E_P = \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \frac{1}{ik\sin\theta}\left[e^{(1/2)ik(-a+b)\sin\theta} - e^{(1/2)ik(-a-b)\sin\theta} + e^{(1/2)ik(a+b)\sin\theta} + e^{(1/2)ik(a-b)\sin\theta} \right]$$

With $\beta = (1/2)kb\sin\theta$ and $\alpha = (1/2)ka\sin\theta$

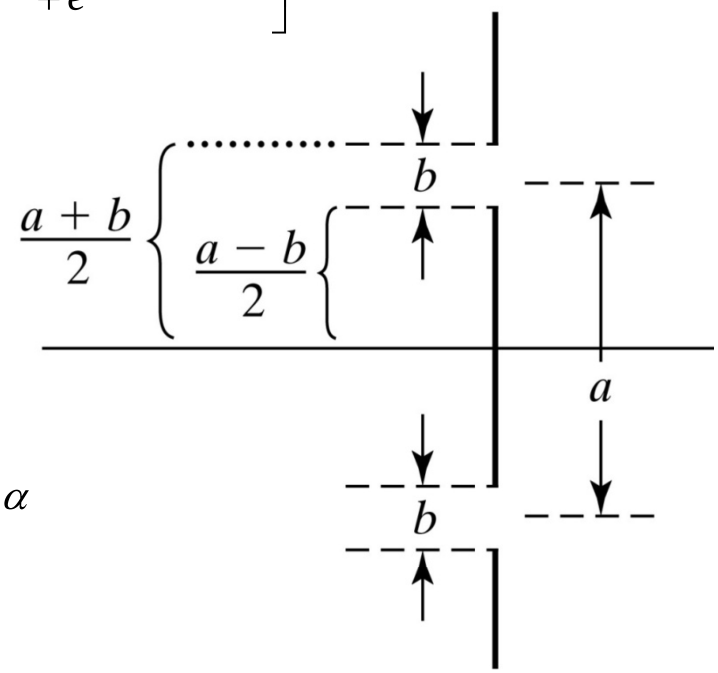
$$E_P = \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \frac{b}{2i\beta}\left[e^{i\alpha}\left(e^{i\beta} - e^{i\beta}\right) + e^{-i\alpha}\left(e^{i\beta} - e^{i\beta}\right) \right]$$

$$E_P = \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \frac{b}{2i\beta}\left[\left(e^{i\alpha} + e^{-i\alpha}\right)\left(e^{i\beta} - e^{i\beta}\right) \right]$$

Using Euler's equation:

$$\mathrm{E_P} = \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \frac{b}{2i\beta}\left[(2\cos\alpha)(2i\sin\beta) \right] = \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \frac{2b\sin\beta}{\beta}\cos\alpha$$

$$E_P = E_0 e^{i(kr_0-\omega t)} \text{ where } E_0 = \frac{E_L}{r_0}\frac{2b\sin\beta}{\beta}\cos\alpha$$



The irradiance at point P is:

$$I = \left(\frac{\varepsilon_0 c}{2}\right) E_0^2 = \left(\frac{\varepsilon_0 c}{2}\right)\left(\frac{2bE_L}{r_0}\right)^2 \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha \rightarrow \boxed{I = 4I_0\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\alpha} \text{ where } \boxed{I_0 = \left(\frac{\varepsilon_0 c}{2}\right)\left(\frac{2bE_L}{r_0}\right)^2}$$

Double-slit diffraction II

With the modified interference pattern for double-slit due to diffraction of each slit we have:

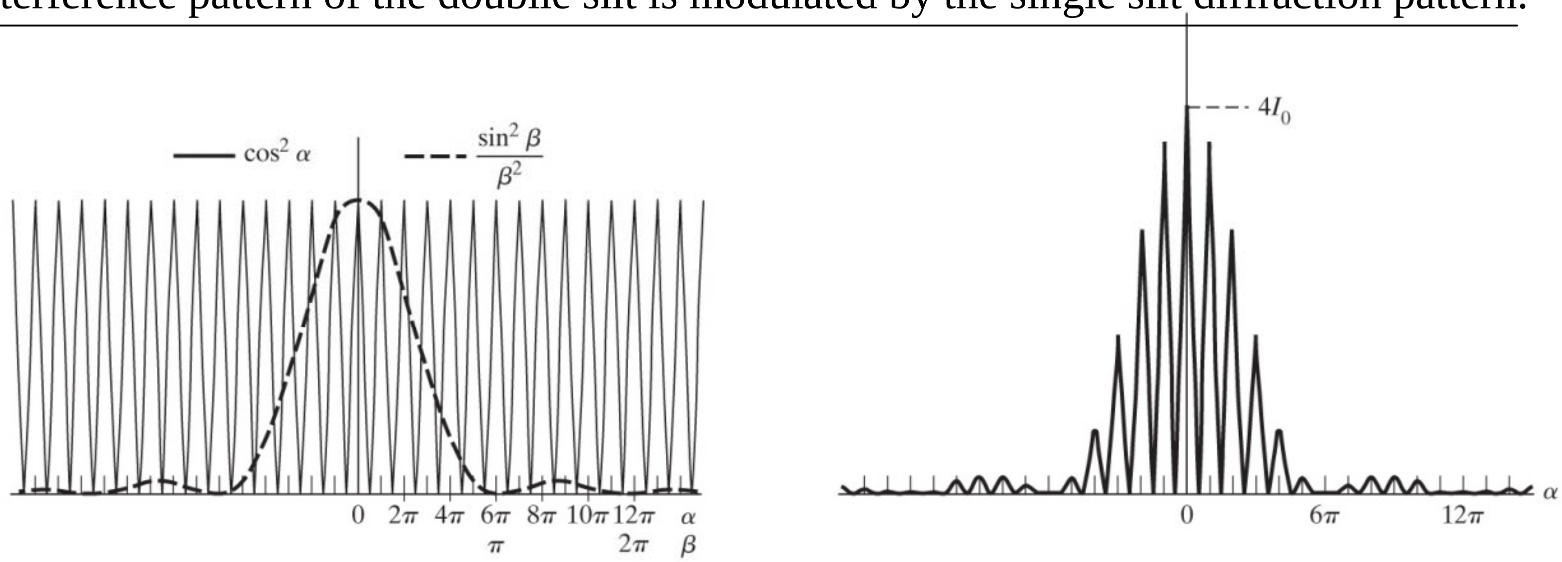
$$I = \underbrace{4I_0\cos^2(\alpha)}_{\substack{\text{Interference pattern}\\ \text{of the double-slit}}}\underbrace{\left(\frac{\sin\beta}{\beta}\right)^2}_{\substack{\text{Diffraction pattern}\\ \text{of a single slit}}} \quad \text{where } I_0 = \left(\frac{\varepsilon_0 c}{2}\right)\left(\frac{2bE_L}{r_0}\right)^2$$

where $\alpha = \dfrac{ka\sin\theta}{2} = \dfrac{\pi a\sin\theta}{\lambda}$ and $\beta = kb\sin\theta$

Now $I_{\max} = 4I_0$ which is expected for the coherent sources.

In figure below a=6b and α=6β thus the cosα varies much more rapidly than sinc$^2\beta$

We say the interference pattern of the doublle slit is modulated by the single slit diffraction pattern.



(a) (b)

Diffraction from many slits I

Diffraction pattern of a plane wavefront that is obstructed everywhere but att the two slits shown in fig. We follow our analysis for the single slit. The diffraction pattern for two slits is going to be superposition of the patterns by each slit. Therefore we write:

$$E_P = \sum_{j=1}^{N/2} \int_{slit(j)} dE_{Pi} = \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \sum_{j=1}^{N/2} \underbrace{\left\{ \int_{[-(2j-1)a-b]/2}^{[-(2j-1)a+b]/2} e^{iks\sin\theta} ds + \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \int_{[(2j-1)a-b]/2}^{[(2j-1)a+b]/2} e^{iks\sin\theta} ds \right\}}_{K}$$

Here we are considering the pairs of slits that are symmetrical with respect to the center of the grating. $J=1,$ double slit, $j=2,$ 4 slits, etc. With $\beta = (1/2)kb\sin\theta$ and $\alpha = (1/2)ka\sin\theta$

$$K = \frac{E_L}{r_0} e^{i(kr_0-\omega t)} \frac{b}{2i\beta}\left[e^{i\alpha}\left(e^{i\beta}-e^{i\beta}\right) + e^{-i\alpha}\left(e^{i\beta}-e^{i\beta}\right)\right]$$

$$K = \frac{b}{2i\beta}(2i\sin\beta)\left(2\cos(2j-1)\alpha\right) = 2b\frac{\sin\beta}{\beta}\mathrm{Re}\left[e^{i(2j-1)\alpha}\right]$$

$$S = \sum_{j=1}^{N/2} 2b\frac{\sin\beta}{\beta}\mathrm{Re}\left[e^{i(2j-1)\alpha}\right] = 2b\frac{\sin\beta}{\beta}\mathrm{Re}\underbrace{\left[e^{i\alpha}+e^{i3\alpha}+e^{i5\alpha}+\ldots+e^{i(N-1)\alpha}\right]}_{\text{Geometric series}}$$

For a geometric series $a + ar + ar^2 + \ldots.. + ar^n = a\left(\dfrac{r^n-1}{r-1}\right)$

first tem $a = e^{i\alpha}$ and ratio $r = e^{2i\alpha}$ then $S = 2b\dfrac{\sin\beta}{\beta}\mathrm{Re}\left[e^{2i\alpha}\left(\dfrac{\left(e^{2i\alpha}\right)^{N/2}-1}{e^{2i\alpha}-1}\right)\right]$

Diffraction from many slits II

Continued fromlast page:

$$e^{2i\alpha}\left(\frac{\left(e^{2i\alpha}\right)^{N/2}-1}{e^{2i\alpha}-1}\right)=\left(\frac{e^{iN\alpha}-1}{e^{i\alpha}-e^{-i\alpha}}\right)=$$

$$\frac{\cos N\alpha+i\sin N\alpha-1}{2i\sin\alpha}=\frac{-\sin \mathrm{N}\alpha+i\left(\cos N\alpha-1\right)}{-2\sin\alpha}$$

$$S=2b\frac{\sin\beta}{\beta}\mathrm{Re}\left[\frac{-\sin \mathrm{N}\alpha+i\left(\cos N\alpha-1\right)}{-2\sin\alpha}\right]\to S=b\frac{\sin\beta}{\beta}\frac{\sin \mathrm{N}\alpha}{\sin\alpha}$$

$$E_P=\frac{E_L}{r_0}e^{i(kr_0-\omega t)}b\left\{\frac{\sin\beta}{\beta}\frac{\sin \mathrm{N}\alpha}{\sin\alpha}\right\} \text{ and } \boxed{I=I_0\underbrace{\left(\frac{\sin\beta}{\beta}\right)^2}_{\text{Diffraction by a single slit}}\underbrace{\left(\frac{\sin \mathrm{N}\alpha}{\sin\alpha}\right)^2}_{\text{Interference between N slits}}}$$

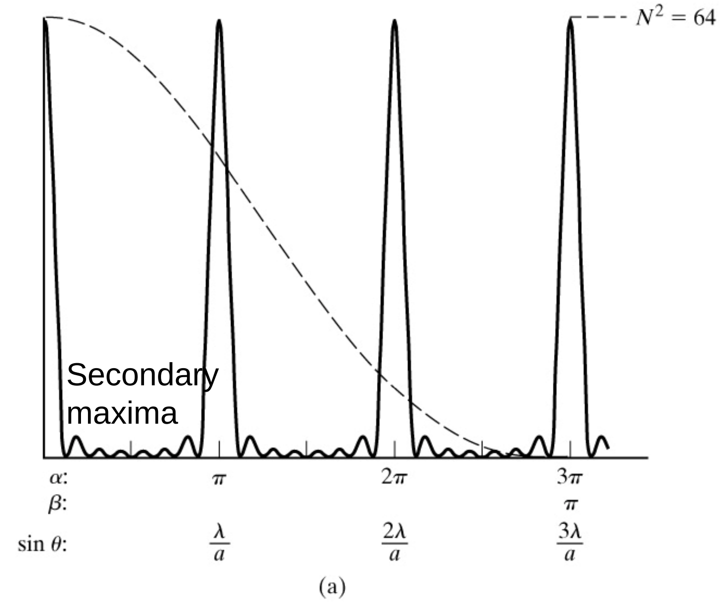
With $\boxed{\beta=(1/2)kb\sin\theta}$ and $\boxed{\alpha=(1/2)ka\sin\theta}$

$\underbrace{\lim_{a\to m\pi}\frac{\sin \mathrm{N}\alpha}{\sin\alpha}}_{\text{indeterminate}}=\lim_{a\to m\pi}\frac{N\cos \mathrm{N}\alpha}{\cos\alpha}=\pm N$ this resembles a series of sharp maxima that we call principal maxima.

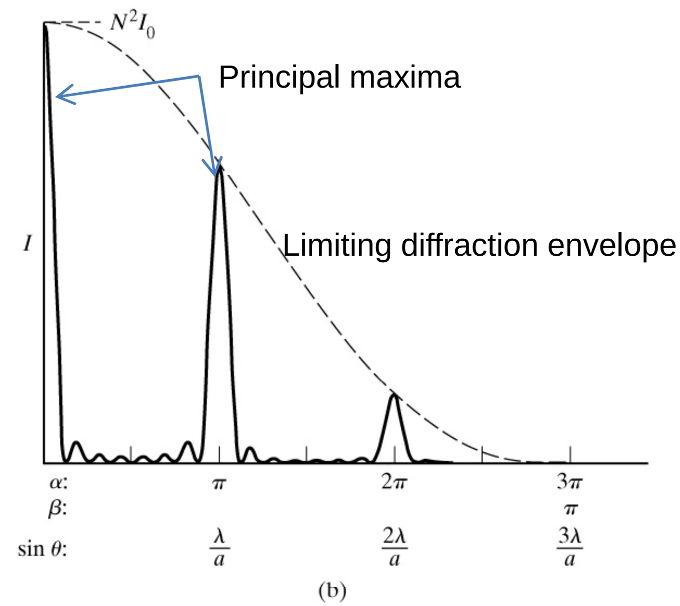
$\boxed{\mathrm{I}_{\text{principal maxima}}\propto N^2}$ cantered at values $\alpha=0,\pm\pi,\pm2\pi,\pm3\pi$

Between successive peaks there are $N-2$ secondary peaks.

The full irradiance is product of the diffraction pattern and interference pattern.

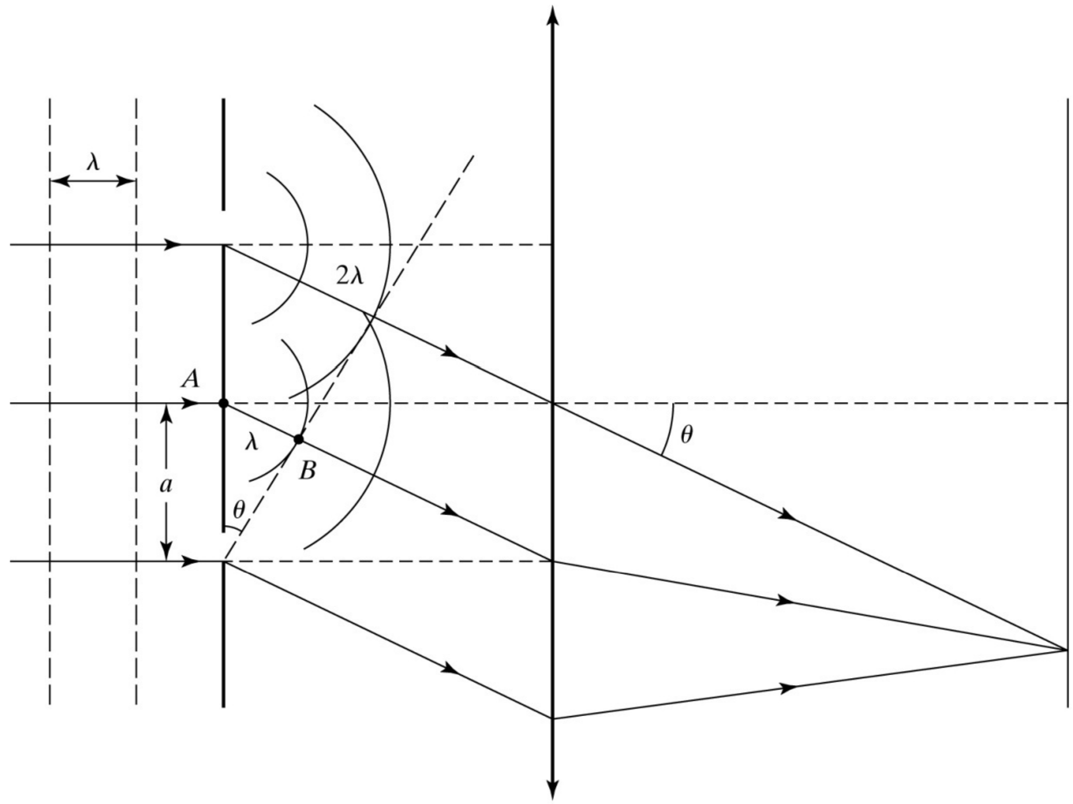


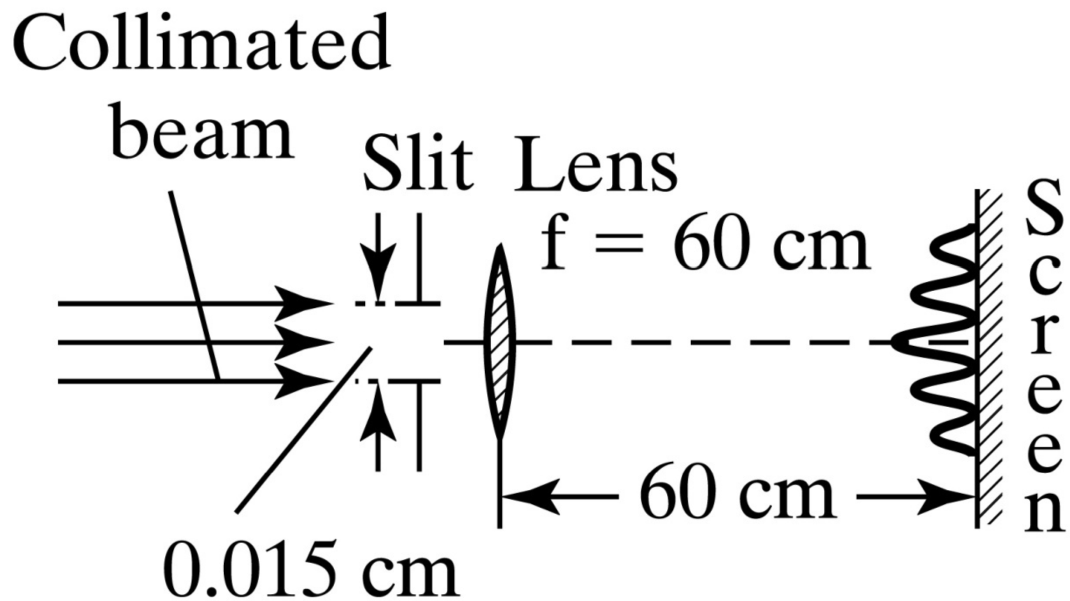
(a)

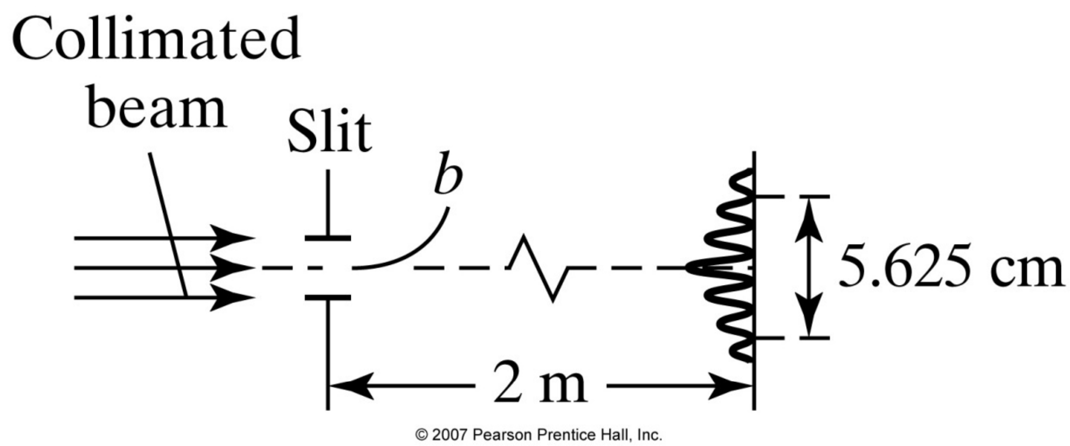


(b)

Formation of secondary maxima


λ
2λ
A
λ
B
a
θ
θ


Collimated
beam
Slit
Lens
f = 60 cm
Screen
0.015 cm
60 cm


Collimated
beam
Slit
b
5.625 cm
2 m
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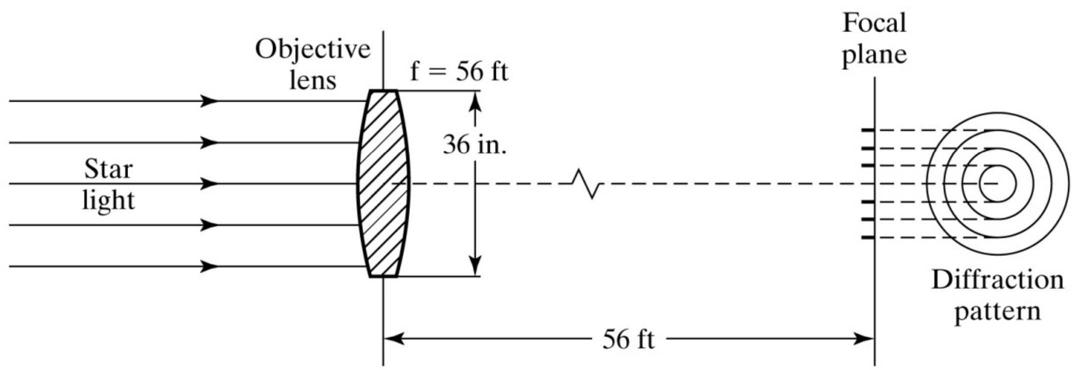
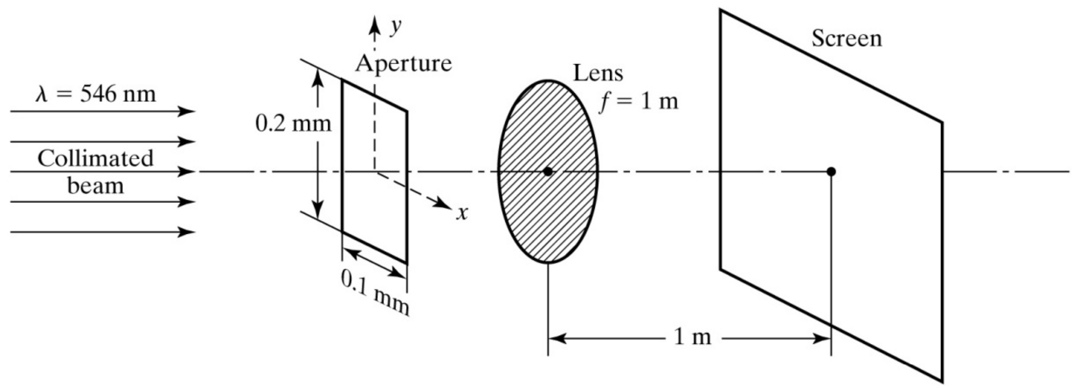

Objective
lens
f = 56 ft
36 in.
Star
light
Focal
plane
Diffraction
pattern
56 ft



λ = 546 nm
Collimated
beam
0.2 mm
y
Aperture
x
0.1 mm
Lens
f = 1 m
Screen
1 m