ME 297
L8 Mirror matrices
Matrix formalism to model reflection from the plane mirrors

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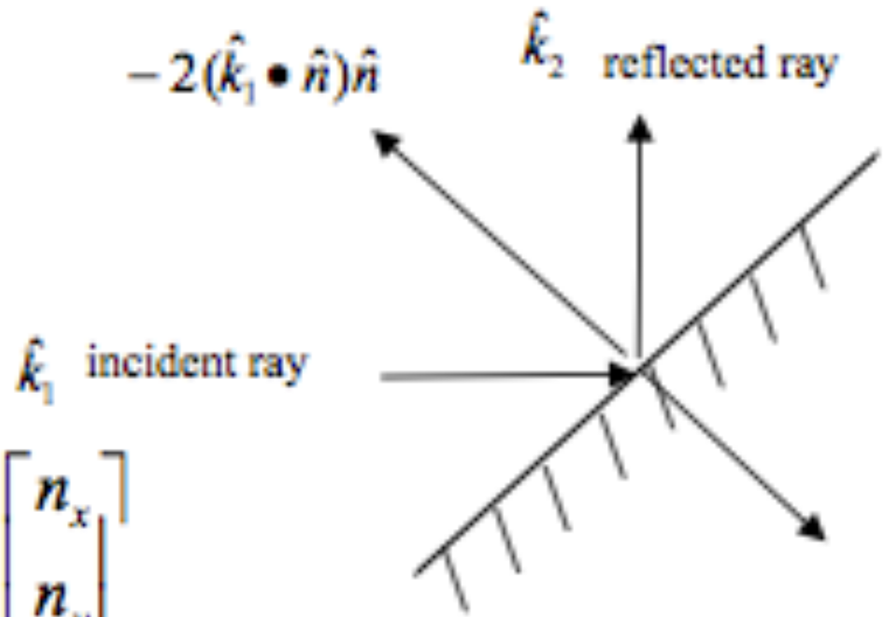
Ref. Dr. Jim Burge’s Notes

Vector form of the law of reflection

$$\hat{k}_2 = \hat{k}_1 - 2(\hat{k}_1 \bullet \hat{n})\hat{n}$$



The hats indicate unit vectors
k_1 = incident ray
k_2 = reflected ray
n = surface normal

$$\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k} \qquad n = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

Matrix form of the law of reflection

$$k_2 = M\ k_1$$

And the mirror matrix becomes:

$$M = I - 2n \cdot n^T$$

Matrix form of the law of reflection

$k_2 = M\,k_1$ where $M = I - 2n \cdot n^T$ Expanding M we get:

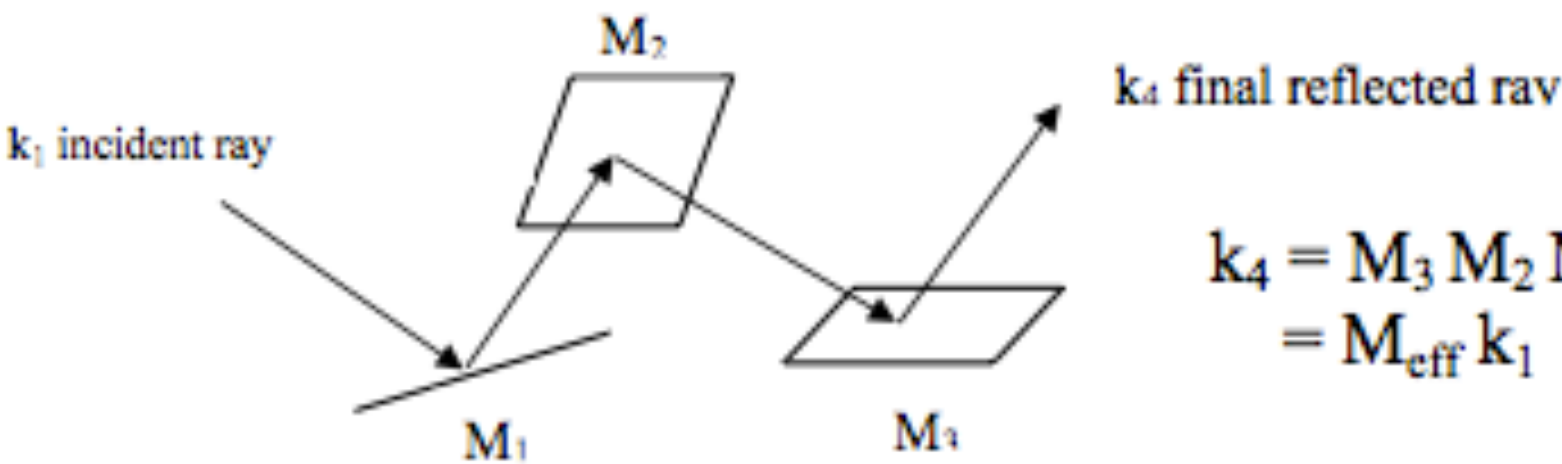
$$M := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \cdot \begin{pmatrix} n_x & n_y & n_z \end{pmatrix}$$

After calculating this mirror matrix, any vector k1 gets changed by reflection from the mirror to a new vector k2, calculated by simple matrix multiplication

$$M := \begin{bmatrix} 1 - 2 \cdot n_x^{\,2} & -2 \cdot n_x \cdot n_y & -2 \cdot n_x \cdot n_z \\ -2 \cdot n_x \cdot n_y & 1 - 2 \cdot n_y^{\,2} & -2 \cdot n_y \cdot n_z \\ -2 \cdot n_x \cdot n_z & -2 \cdot n_y \cdot n_z & 1 - 2 \cdot n_z^{\,2} \end{bmatrix}$$

A series of reflections

- A series of reflections is modeled by successive mirror matrix multiplications.
- Effect of any set of mirrors can be reduced to a single 3x3 matrix multiplication.
- If light bounces off mirror 1, then 2 then 3, the net effect of these three reflections is



$$k_4 = M_3 M_2 M_1 k_1$$
$$= M_{eff} k_1$$

Example: find the reflected coordinates by a mirror with its normal in +z direction

For this mirror

$$n = 0\hat{i} + 0\hat{j} + 1\hat{k}$$

and M is: $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$M := \begin{bmatrix} 1 - 2 \cdot n_x^2 & -2 \cdot n_x \cdot n_y & -2 \cdot n_x \cdot n_z \\ -2 \cdot n_x \cdot n_y & 1 - 2 \cdot n_y^2 & -2 \cdot n_y \cdot n_z \\ -2 \cdot n_x \cdot n_z & -2 \cdot n_y \cdot n_z & 1 - 2 \cdot n_z^2 \end{bmatrix}$$

And a set of coordinates will be reflected as

$$x' = Mx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$y' = My = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad z' = Mz = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

An incident ray traveling in the +z direction will be reflected to travel in the –z direction. Images of the x and y axes do not change.

Parity

- The **determinant of the mirror matrix gives the parity of the system.**
- An even number of reflections will cause the image to be right-handed, or to have parity = det(M) = 1.
- A system with an odd number of reflections will cause the image to be left-handed, or to have parity = det(M) = -1.
- Parity of one mirror is odd (–1).
- The image of a right handed coordinate system will appear to be left-handed in the reflection.
- This means that clockwise rotation about any basis vector will appear counter-clockwise in the image.

Rotating a mirror

- Mirrors with any orientation can be defined using rotations. The effect of rotating a mirror M, or system of mirrors that has equivalent matrix M is

$$M_r = R \cdot M \cdot R^T$$

where M_r is the new matrix and R is the rotation matrix given below

x rotation $$R_x := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

y rotation $$R_y := \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

z rotation $$R_z := \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: rotation of a mirror about x axis with its normal in +z direction

Rotation about x-axis by α: the mirror matrix is

$$M_r = R_x(\alpha) M_z R_x(\alpha)^T$$

Using the identity: $[AB]^T = B^T A^T$ we have:

$$M_r = R_x(\alpha) M_z R_x(\alpha)^T = R_x(\alpha)\left[R_x(\alpha) M_z^T\right]^T$$

For this special case: $M_z^T = M_z$ & $\left[R_x(\alpha) M_z\right] = \left[R_x(\alpha) M_z\right]^T$

$$M_r = R_x(\alpha)\left[R_x(\alpha) M_z^T\right]^T = R_x(\alpha) R_x(\alpha) M_z = R_x(2\alpha) M_z$$

So effect of α-rotation around x is $2\alpha - rotation$ on the reflected beam.

Likewise effect of β-rotation around y is 2β-rotation on the reflected beam.

Exercise

Show that effect of γ-rotation around the z-axis on a mirror matrix wih its normal on the +z direction is:

$$M_r = R_z(\gamma) M_z R_z^T(\gamma) = M_z$$

Some common types of mirrors I

Free space

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

x mirror

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

insensitive to x rotation
2θ for y and z rotations

y mirror

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

insensitive to y rotation
2θ for x and z rotations

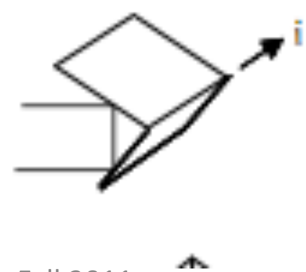
z mirror

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

insensitive to z rotation
2θ for x and y rotations

90° x roof

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



insensitive to x rotation
2θ for y and z rotations

Some common types of mirrors II

90° y roof

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

insensitive to y rotation
2θ for x and z rotations

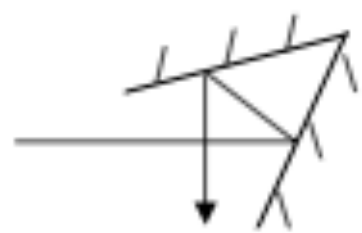
90° z roof

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

insensitive to z rotation
2θ for x and y rotations

45° x roof

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$



90° deviation
insensitive to x rotation
θ for y and z rotations

cube corner

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

retro-reflects
insensitive to all rotations

How to find the mirror matrix for a prism or mirror system

- In most cases it is possible to **find mirror matrix** by **inspection**.
- Following the z, y, and z coordinates of a unit vector through the system using the **bouncing pencil method** and mark their changes as the vector reflects or travels in the system.
- We only need to trace two axes through and use the **parity** to get the third.
- Reverse the direction of rotation if the system has –1 parity
- Each reflection changes parity of the system by -1.

Effect of small rotations of any prism

*A*pply the rotation transformations to the prism matrix *Mp*

$$M_r = R_x(\alpha) M_P R_x^T(\alpha)$$

This matrix defines **the new line of sight and any image rotation**.

For **small angles** (jitter) we use the small angle approximation.

$$\sin\alpha \approx \alpha$$

$$\cos\alpha \approx 1$$

Effect of small rotations of any prism

Some hints on rotation of prisms :

for nearly all cases, prism rotation θ about the x,y,or z axis does one of three things:

1. causes image rotation about same axis by an amount 2θ
2. has no effect on image about any axes
3. causes image to rotate anamount $\pm\theta$ aout the other two axes.