Fresnel and Fraunhofer Diffraction

4.1 Background

- In chapter 3 we dealt with most general form of the diffraction theory.
- In chapter 4 we will deal with
  - Intensity of a wave field
  - Huygens-Fresnel principle
  - Certain approximations to reduce the problem to a simpler mathematical form. These approximations are:
    - Fresnel
    - Fraunhofer
  - We consider the wave propagation phenomenon as a system.
  - The approximations will be valid for certain class of inputs.
  - Preparation for the calculations related to the approximations

4.1.1 The intensity of a wave field

Intensity is the physically measurable attribute of an optical wavefield

Intensity and power density are not the same but proportional

Intensity of a scalar monochromatic wave at point P

$$I(P) = |U(P)|^2$$

For a narrow-band (not perfectly monochromatic) intensity is given by

$$I(P) = \underbrace{\left\langle |u(P,t)| \right\rangle^2}_{\text{An infinite time average}} \quad \text{and} \quad I(P,t) = \underbrace{|u(P,t)|}_{\text{Instantaneous Intensity}}{}^2$$

In calculating a diffraction pattern, we are looking for the intensity of the pattern.

4.1.2 The Huygens-Fresnel principle in rectangular coordinates

According to the first Rayleigh-Somefeld solution the diffracted field on the xy plane due to the aperture on the $\xi\eta$ plane is $U_I\left(P_0\right)$ and the Huygens-Fresnel principle can be written as:

$$U_I\left(P_0\right)=\frac{1}{j\lambda}\iint_\Sigma U(P_1)\frac{e^{jkr_{01}}}{r_{01}}\cos\theta ds \text{ where } \cos\theta=\cos(\vec{n},\vec{r}_{01})=\frac{z}{r_{01}}$$

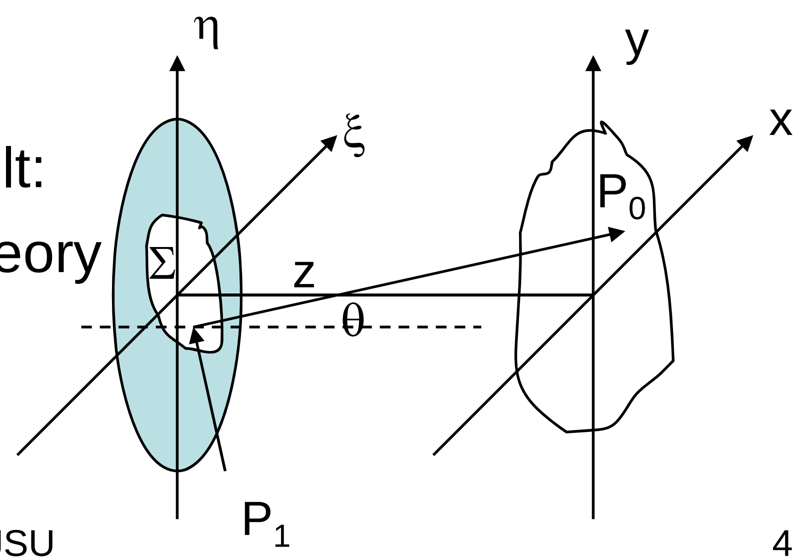
$$U\left(x,y\right)=\frac{z}{j\lambda}\iint_\Sigma U(\xi,\eta)\frac{e^{jkr_{01}}}{r_{01}^2}d\xi d\eta,$$

where $r_{01}=\sqrt{z^2+(x-\xi)^2+(y-\eta)^2}$

Two approximations are used in this result:

1) inherent approximation in the scalar theory

2) $r_{01}>>\lambda$



4.2 The Fresnel approximation I

Goal: to reduce the H-F principle to a simple and usable expression

We achieve this by approximations for r_{01} :

Binomial expansion: $(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + ... + y^n,\ n = 1,2,3,...$

$(1+b)^{\pm 1/2} = 1 \pm \frac{1}{2}b \mp \frac{1}{8}b^2 \pm ...$ for $-1 < b \le 1$: the higher order terms are negligable

$$r_{01} = z\sqrt{1+\left(\frac{x-\xi}{z}\right)^2+\left(\frac{y-\eta}{z}\right)^2} \approx z\left[1+\frac{1}{2}\left(\frac{x-\xi}{z}\right)^2+\frac{1}{2}\left(\frac{y-\eta}{z}\right)^2+...\right]$$

Where do we cut the series? We will use r_{01} in the diffracted field equation

$$U(x,y) = \frac{z}{j\lambda}\iint_\Sigma U(\xi,\eta)\frac{e^{jkr_{01}}}{r_{01}^2}d\xi d\eta$$

The term $e^{jkr_{01}}$ is very sensitive to the values of r_{01} specially since it is multiplied by a very large number $k = 2\pi/\lambda$. In the visible of the order 10^7. We keep two terms for the exponent. For r_{01}^2 error introduced by dropping all terms but z is small.

$$U(x,y) = \frac{z}{j\lambda}\iint_\Sigma U(\xi,\eta)\frac{e^{jkr_{01}}}{r_{01}^2}d\xi d\eta = \frac{e^{jkz}}{j\lambda z}\int\int_{-\infty}^{\infty} U(\xi,\eta)e^{\left\{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}}d\xi d\eta$$

The integration limit is let to ∞ using the usual boundary conditions.

4.2 The Fresnel approximation II

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}\int\int_{-\infty}^{\infty}U(\xi,\eta)e^{\left\{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}}d\xi d\eta$$ this looks like a convolution

$$\underbrace{U(x,y)=\int\int_{-\infty}^{\infty}U(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta}_{\text{First form of the Fresnel diffraction integral}}$$ where $h(x,y)=\frac{e^{jkz}}{j\lambda z}e^{\left\{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}}$

Another form of the Fresnel diffraction integral is expressed as the following

$$\underbrace{U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\overbrace{\int\int_{-\infty}^{\infty}\left\{U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta}^{\text{Fourier transform of the } U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)} \text{ which is complex field just to the right of aperture multiplied by a quadratic phase factor}}}_{\text{Seond form of the Fresnel diffraction integral}}$$

Observation in the near field of the aperture or Fresnel diffraction region

Where $r_{01}>>\lambda,\ \frac{x-\xi}{z}<1,\ \frac{y-\eta}{z}<1,$

and scalar theory approximation are assumed

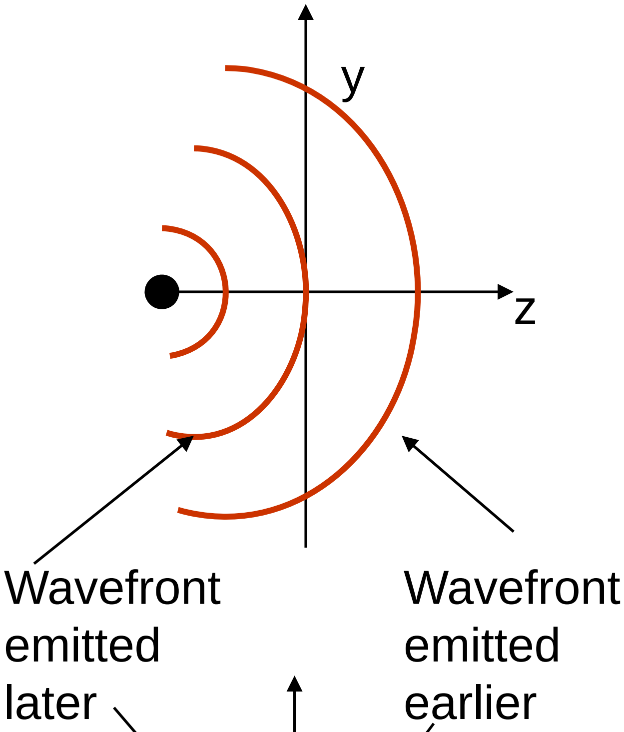
4.2.1 Positive vs. negative phases

Goal: to understand the meaning of the signs of the phase exponentials:

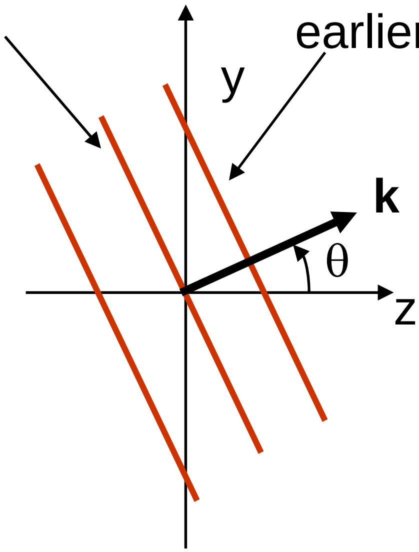
in the spherical wave $e^{jkr_{01}}$ and its equivalent

in the quadratic approximation $e^{j\frac{k}{2z}(x^2+y^2)}$ (for $z > 0$)

Sign convention: our phasors rotate in the clockwise direction (the angle becomes more negative as time goes) and their time dependence is $e^{-j2\pi\nu t}$



We move in space in such a way that we encounter portions of the wavefield that were emitted earlier in time. The phase must become more positive since the phasor had not have time to rotate as far in clockwise.



We move in space in such a way that we encounter portions of the wavefield that were emitted later in time. The phasor will have advanced in the clockwise direction, therefore the phase must become more negative.

4.2.2 Accuracy of the Fresnel approximation I

Fresnel approximation replaced the spherical secondary wavelets with parabolic wavefronts in the Huygens-Fresnel principle

$$U(x,y)=\frac{z}{j\lambda}\iint_{\Sigma}U(\xi,\eta)\underbrace{\frac{e^{jkr_{01}}}{r_{01}^2}}_{\text{Spherical wavelets}}d\xi d\eta \rightarrow \frac{e^{jkz}}{j\lambda z}\int\int_{-\infty}^{\infty}U(\xi,\eta)\underbrace{e^{\left\{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}}}_{\text{Parabolic wavelets}}d\xi d\eta$$

The accuracy of this approximation depends on the size of the higher order terms in binomial expansion. A sufficient condition for accuracy is:

$$r_{01}\approx z\left[1+\frac{1}{2}\left(\frac{x-\xi}{z}\right)^2+\frac{1}{2}\left(\frac{y-\eta}{z}\right)^2+\underbrace{\frac{1}{8}\left\{\left(\frac{x-\xi}{z}\right)^2+\left(\frac{y-\eta}{z}\right)^2\right\}^2}_{\substack{\text{maximum phase change due to dropping}\\ b^2/8 \text{ term must be much less than one radian}}}\quad\ldots\right]$$

$$e^{j\overbrace{kr_{01}}^{\phi}}\rightarrow\Delta\phi(O^2)=\frac{2\pi}{\lambda}\frac{1}{8}\left\{\left(\frac{x-\xi}{z}\right)^2+\left(\frac{y-\eta}{z}\right)^2\right\}^2_{Max}<<1$$

$$z^3>>\frac{\pi}{4\lambda}\left\{(x-\xi)^2+(y-\eta)^2\right\}^2_{Max}$$

4.2.2 Accuracy of the Fresnel approximation II

Example: calculate the safe distance to use the Fresnel approximation for a circular aperture of size $1cm$ and a circular observation ragion of $1cm$ with a light of $\lambda = 0.5\mu m$. (Answere: $z >> 25\ cm$)

$z^3 >> \frac{\pi}{4\lambda}\left\{(x-\xi)^2 + (y-\eta)^2\right\}^2_{Max}$ hint: $x-\xi$ and $y-\eta$ should have their maximum possible values to evaluate the condition.

If the higher order terms do not change the value of the Fresnel integral substantially, we can use the approximation.

They do not need to be small in this case.

4.2.3 The Fresnel approximation and the angular spectrum I

Goal: understand the implications of the Fresnel approximations from the point of view of angular spectrum method of analysis.

We compare the transfer function of propagation through free space, predicted by RS scalar diffraction theory, with the transfer function predicted by the Fresnel analysis

$$H_{RS}(f_X, f_Y) = \begin{cases} \overbrace{e^{\left(j\frac{2\pi z}{\lambda}\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\right)}}^{\text{General spatial phase dispersion representing propagation}} & \sqrt{f_X{}^2 + f_Y{}^2} < 1/\lambda \\ 0 & \text{otherwise} \end{cases}$$

← RS theory

$$\underbrace{h(x,y) = \frac{e^{jkz}}{j\lambda z} e^{\left\{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}}}_{\text{Fresnel approximation impulse response}} \xrightarrow{\text{FT}} H_F(f_X, f_Y) = \overbrace{\underbrace{e^{jkz}}_{\text{All plane waves suffer equally}}}^{\text{A constant phase delay due to traveling}} \; \overbrace{\underbrace{e^{-j\pi\lambda z\left(f_X{}^2+f_Y{}^2\right)}}_{\text{Different plane-wave components suffer different phase delays}}}^{\text{Qadratic phase dispersion}}$$

4.2.3 The Fresnel approximation and the angular spectrum II

$$H_{RS}(f_X,f_Y)=\begin{cases} e^{\left(j\frac{2\pi z}{\lambda}\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\right)} & \sqrt{f_X{}^2+f_Y{}^2}<1/\lambda \\ 0 & \text{otherwise}\end{cases}$$ ← RS theory

$$H_F(f_X,f_Y)=e^{jkz}e^{-j\pi\lambda z\left(f_X{}^2+f_Y{}^2\right)}$$

We can see that $H_F(f_X,f_Y)$ is an approximation to the $H_{RS}(f_X,f_Y)$

Applying the binomial expansion to the $H_{RS}(f_X,f_Y)$ we get:

$$\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\approx 1-\frac{(\lambda f_X)^2}{2}-\frac{(\lambda f_Y)^2}{2}$$ if $(\lambda f_X)^2 \ll 1$ *and* $(\lambda f_Y)^2 \ll 1$

$$e^{\left(j\frac{2\pi z}{\lambda}\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\right)}\approx e^{\left(j\frac{2\pi z}{\lambda}\left[1-\frac{(\lambda f_X)^2}{2}-\frac{(\lambda f_Y)^2}{2}\right]\right)}=e^{jkz}e^{-j\pi\lambda z\left(f_X{}^2+f_Y{}^2\right)}=H_F(f_X,f_Y)$$

Conclusion:

$H_{RS}(f_X,f_Y)\approx H_F(f_X,f_Y)$ When the conditions: $(\lambda f_X)^2=\alpha^2=\left(\frac{k_x}{|k|}\right)^2\ll 1,$

$(\lambda f_Y)^2=\beta^2=\left(\frac{k_Y}{|k|}\right)^2\ll 1$ are satisfied. So Fresnel approxilation is equivalent to the paraxial approximation that is limitted to small propagation angles.

4.2.4 Fresnel diffraction between confocal spherical surfaces I

Goal: analysis of diffraction between two confocal spherical surfaces

Confocal spheres: center of each lies on the surface of the other.

We set the spheres tangant to the plance we used before.

Located at $z = 0$ and $z = z$. r_{01} is the distance between two spherical caps.

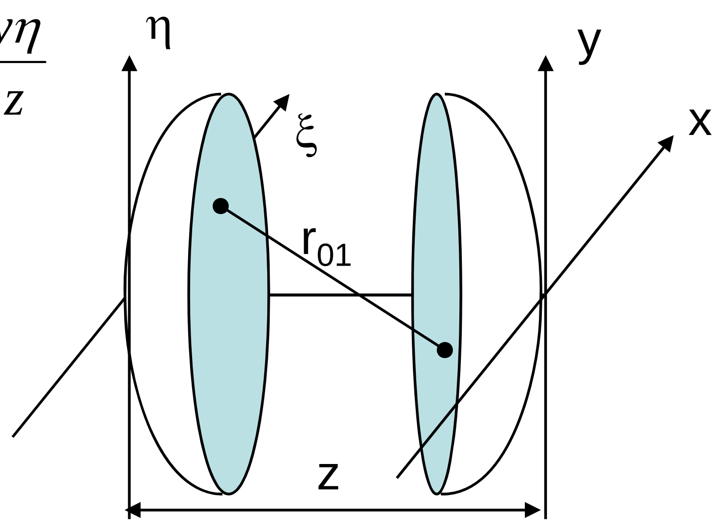
We write the equations for both surfaces and find the distance between them. Make paraxial approximation by using the binomial expansion.

Assuming the extend of the spherical caps about the z axis is small compared to the radii of the spheres, i.e. for z>> $x - \xi$ and $y - \eta$, we get

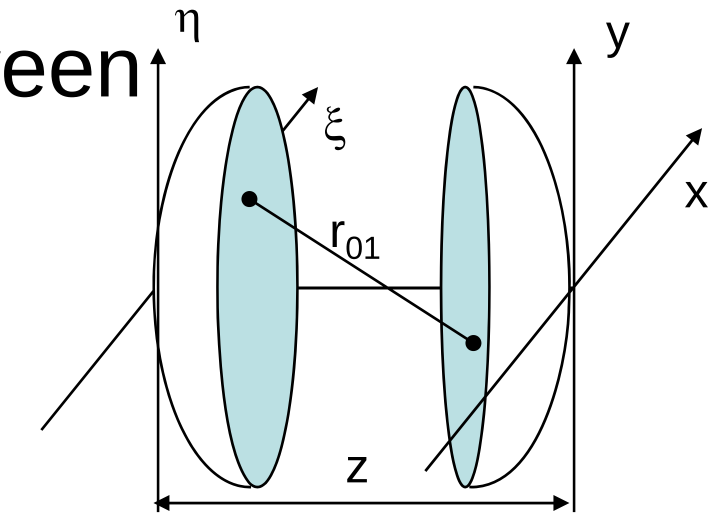
$$r_{01} = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2} \;\rightarrow\; r_{01} \approx z - \frac{x\xi}{z} - \frac{y\eta}{z}$$

$$U(x,y) = \frac{z}{j\lambda} \iint_{\Sigma} U(\xi,\eta) \frac{e^{jkr_{01}}}{r_{01}^2} d\xi d\eta,$$

$$\underbrace{U(x,y)}_{\text{Field on the right hand spherical cap}} = \frac{e^{jkz}}{j\lambda z} \underbrace{\int \int_{-\infty}^{\infty} U(\xi,\eta) e^{\left\{-j\frac{2\pi}{\lambda z}[x\xi + y\eta]\right\}} d\xi d\eta}_{\text{Fourier transform of the Field on the left hand spherical cap}}$$



4.2.4 Fresnel diffraction between confocal spherical surfaces II



Comparing the two form:

$$\underbrace{U(x,y)}_{\text{Field on the right hand spherical cap}} = \frac{e^{jkz}}{j\lambda z}\underbrace{\int\int_{-\infty}^{\infty} U(\xi,\eta)e^{\left\{-j\frac{2\pi}{\lambda z}[x\xi+y\eta]\right\}}d\xi d\eta}_{\text{Fourier transform of the Field on the left hand spherical cap}}$$

Compared with the Fresnel diffractionintegral:

$$\underbrace{U(x,y) = \frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\overbrace{\int\int_{-\infty}^{\infty}\left\{U(\xi,\eta)e^{j\frac{k}{2z}(\xi^2+\eta^2)}\right\}e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta}^{\text{Fourier transform of the } U(\xi,\eta)e^{j\frac{k}{2z}(\xi^2+\eta^2)} \text{ which is complex field just to the right of aperture multiplied by a quadratic phase factor}}}_{\text{Seond form of the Fresnel diffraction integral}}$$

We see that by replacing the two plates with spherical caps, the quadratic factor in (x, y), $e^{j\frac{k}{2z}(x^2+y^2)}$, and (ξ,η), $e^{j\frac{k}{2z}(\xi^2+\eta^2)}$, have been elminated.

In fact these two phase factors are paraxial representations of spherical phase surfaces. By having a spherical observation plane, they are gone.

On derivation of the Fresnel diffraction integral we approximated the spherical waves with plane waves. Now getting back to spherical surfaces, there is no approximation. Sherical surface will see the spherical wave like a flat surface sees the plane wave.

4.3 The Fraunhofer approximation I

Goal: applying another more stringent approximation to the Fresnel diffraction integral to simplify the calculations for valid cases.

Fourier transform of the quadratic phase function, $U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}$, which is the aperture distribution $U(\xi,\eta)$, multiplied by a quadratic phase factor $e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}$

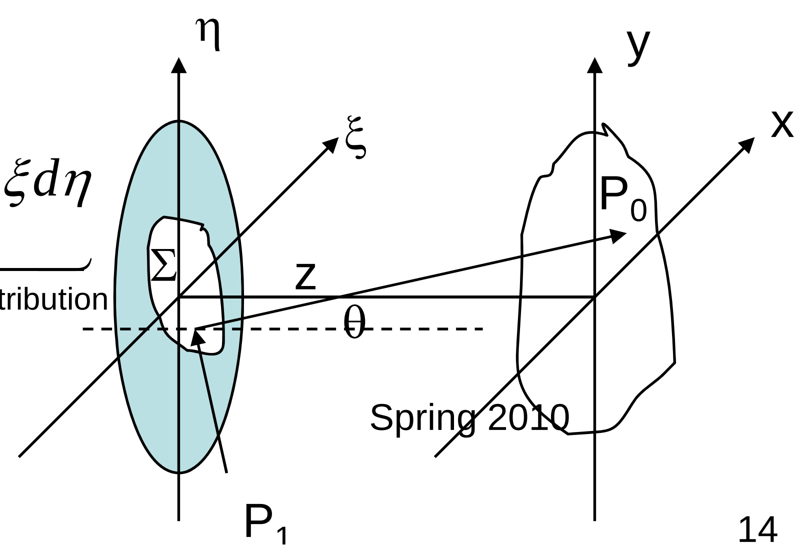
$$U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\int\int_{-\infty}^{\infty}\left\{U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta$$

Seond form of the Fresnel diffraction integral

Applying the Fraunhofer approximation: $z \gg \frac{k\left(\xi^2+\eta^2\right)_{\max}}{2}$ the quadratic phase factor $e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)} \approx 1$

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\int\int_{-\infty}^{\infty}U(\xi,\eta)e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta$$

Fourier transform of the aperture distribution evaluated at $f_X=\frac{x}{\lambda z}$ and $f_Y=\frac{y}{\lambda z}$



4.3 The Fraunhofer approximation II

Fraunhofer approximation: $z \gg \frac{k\left(\xi^2+\eta^2\right)_{\max}}{2}$ or $\frac{z\lambda}{\pi} \gg$ aperture size,

Fresnel approximation: $z \gg \sqrt{(x-\xi)^2+(y-\eta)^2}$

Since $k=\frac{2\pi}{\lambda}$ is a large number Fraunhofer approximation is much stringent than Fresnel approximation

At optical frequencies: $\lambda = 0.6\mu m;\ aperture\ width = 2.5\ cm;\ z \gg 1600m$

A less stringent condition is called the "antenna designer formula":

for an aperture with linear dimension of D, the Fraunhofer approximation will be valid if $z > \frac{2D^2}{\lambda}$ now $z > 2000$ meters (>> is replaced with >)

The Frauhofer diffraction pattern will form at very far distances but we can bring the pattern by using a proper lens or proper ilumination.

Will see in the problems.

Bessel functions I

The Bessel functions or cylinder functions or cylinderical harmonics of the first kind, $J_n(x)$, are defined as the solutions to the

Bessel differential equation: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$

These functions are nonsingular at the origin.

$$J_m(x)\begin{cases} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+|m|} l!(|m|+l)!} x^{2l+|m|} & |m| \neq \frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \sin x & m = \frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \cos x & m = -\frac{1}{2} \end{cases}$$

$$J_{-m}\left(x\right) = (-1)^m J_m(x) \quad m = 0, 1, \ 2, \ 3, \ \ldots$$

A derivative identity: $\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x)$

An integral identity: $\int_0^u u' J_0(u')du' = uJ_1(u)$

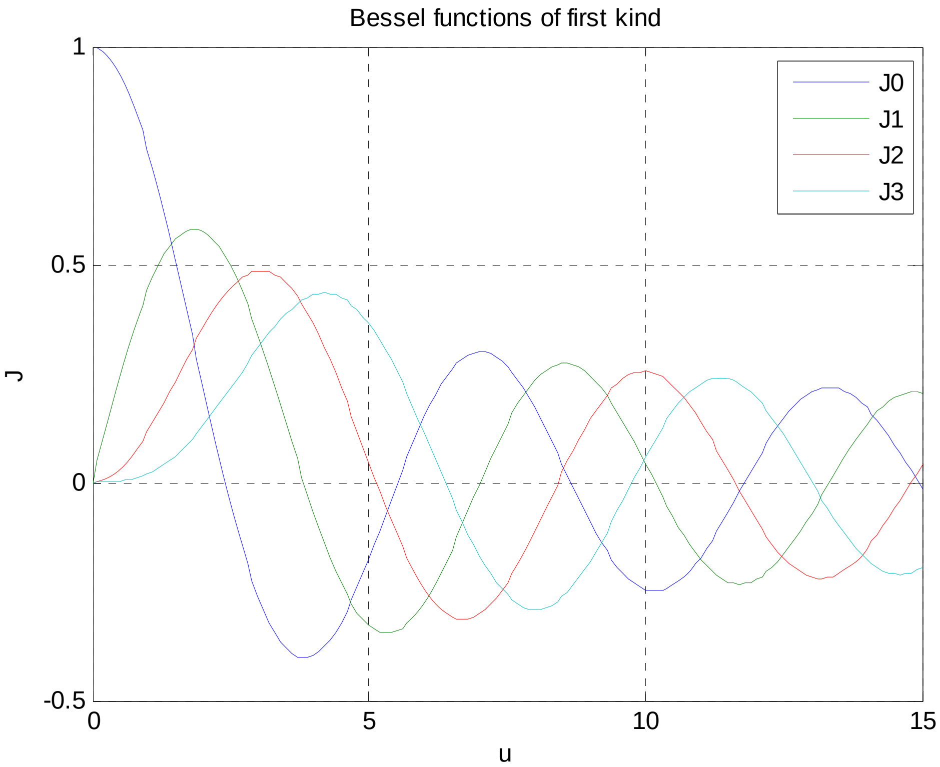
Bessel function addition theorem: $J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y) J_{n-m}(z)$

$\sum_{k=-\infty}^{\infty} J_k(x) = 1; \quad e^{iz\cos\theta} = J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos(n\theta)$

There are more of these identities. Check you favorite math handbook.

Bessel functions of the first kind (MATLAB)

```
u = (0:0.1:15)
BJ0=besselj(0,u);
BJ1=besselj(1,u);
BJ2=besselj(2,u);
BJ3=besselj(3,u)
plot(u,BJ0,u,BJ1,u,BJ2,u,BJ3);
legend('J0','J1','J2','J3')
title('Bessel functions of first kind');
xlabel('u'),ylabel('J')
grid on
```



Bessel functions II

Various integrals expressed in terms of the Bessel functions:

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\ sin\theta - n\theta)d\theta$$ Bessel's first integral

$$J_n(z) = \frac{i^{-n}}{\pi}\int_0^{\pi} e^{iz\cos\theta} \cos(n\theta)d\theta$$

$$J_n(z) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{iz\cos\theta}e^{in\theta}d\theta \to$$ with $n = 0$

$$J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{ia\cos\theta}d\theta$$ or $$J_0(z) = \sum_{k=0}^{\infty}(-1)^k \frac{\left(Z^2/4\right)^k}{\left(k!\right)^2}$$

$$J_n(z) = \frac{2}{\pi}\frac{z^n}{\left(2n-1\right)!!}\int_0^{\pi/2} \sin^{2n} u \cos(z\cos u)du$$ for $n = 1, 2, ...$

$$J_n(x) = \frac{1}{2\pi i}\int_{\gamma} e^{\frac{x}{2}\frac{z-1}{z}} z^{-n-1}dz \quad for\ n > -\frac{1}{2}$$

The Bessel functions are normaized: $\int_0^{\infty} J_n(x)dx = 1$ for $n = 0, 1, 2, ...$

Integrals involving $J_1(x)$: $\int_0^{\infty}\left[\frac{J_1(x)}{x}\right]^2 dx = \frac{4}{3\pi}$ and $\int_0^{\infty}\left[\frac{J_1(x)}{x}\right]^2 xdx = \frac{1}{2}$

Fourier transform of a circularly symmetric function I

Most apertures and lenses have circular symmetry for example

$g(x,y)=\{\begin{smallmatrix}1 & \sqrt{x^2+y^2}\le a \\ 0 & \sqrt{x^2+y^2}>a\end{smallmatrix}$ expresses a circular aperture with radius of a.

The circular symmetry justifies usage of cylindical coordinates.

$$x=r\cos\theta;\quad y=r\sin\theta;\quad r=\sqrt{x^2+y^2};\quad \theta=\tan^{-1}(y/x)$$

$$f_x=\rho\cos\phi;\quad f_y=\rho\sin\phi;\quad \rho=\sqrt{f_X^2+f_Y^2};\quad \phi=\tan^{-1}(f_Y/f_X)$$

$$dxdy=rdrd\theta;\quad df_X df_Y=\rho d\rho d\phi;$$

$$\mathcal{F}\{g(x,y)\}=G(f_X,f_Y)=\iint_{-\infty}^{\infty}g(x,y)e^{-j2\pi(f_X x+f_Y y)}dxdy$$

Now apply change of varaibles:

$$\mathcal{F}\{g(r,\theta)\}=G_0(\rho,\phi)=\int_0^{2\pi}d\theta\int_0^{\infty}g(r,\theta)e^{-j2\pi(\rho\cos\phi r\cos\theta+\rho\sin\phi r\sin\theta)}rdr$$

For circularly symmetric functions g is only function of r. So we write:

$$g(r,\theta)=g_R(r)$$

$$G_0(\rho,\phi)=\int_0^{2\pi}d\theta\int_0^{\infty}g_R(r)e^{-j2\pi r\rho\cos(\theta-\phi)}rdr=\int_0^{\infty}g_R(r)rdr\int_0^{2\pi}e^{-j2\pi r\rho\cos(\theta-\phi)}d\theta$$

Fourier transform of a circularly symmetric function II

$$G_0(\rho,\phi)=\int_0^{\infty} g_R(r)rdr\int_0^{2\pi} e^{-j2\pi r\rho\cos(\theta-\phi)}d\theta$$

this relation is correct for any value of ϕ including $\phi=0$,

Value of the integral $\frac{1}{2\pi}\int_0^{2\pi} e^{-ja\cos(\theta)}d\theta \; = J_0(a)$ is own known as the zeroth order Bessel function of the first kind.

With substituting $a=2\pi r\rho$ and $\phi=0$ we get:

$$\mathcal{B}(\rho)=G_0(\rho)=2\pi\int_0^{\infty} rg_R(r)J_0(2\pi r\rho)dr \leftarrow$$ Fourier-Bessel transform, $\mathcal{B}$, or Hankel transform of zero order

The inverse Fourier-Bessel transform is then:

$$\mathcal{B}^{-1}g(r,\theta)=g_R(r)=2\pi\int_0^{\infty} \rho G_0(\rho)J_0(2\pi r\rho)d\rho$$

Conclusions:

1) Fourier transform of a circularly symmetric function is a circularly summetric function itself.

2) There is no difference between the direct and inverse transform operations.

Fourier transform of a circularly symmetric function III

Following the Fourier integral theorem. and simmilarity theorem, we get:

$\mathcal{B}\mathcal{B}^{-1}\left\{g_R(r)\right\} = \mathcal{B}^{-1}\mathcal{B}\left\{g_R(r)\right\} = \mathcal{B}\mathcal{B}\left\{g_R(r)\right\} = g_R(r)$ $\leftarrow$ when $g_R(r)$ is continuous.

$$\mathcal{B}\left\{g_R(ar)\right\} = \frac{1}{a^2} G_0\left(\frac{\rho}{a}\right)$$

$\mathcal{B}$ for Fourier-Bessel transform.

All other Fourier transform theorems apply since this is just a special case of the general two-dimensional Fourier transforms.

Fourier transform of a circular aperture with radius a

$$g(x,y)=\begin{cases}1 & \sqrt{x^2+y^2}\le a\\ 0 & \sqrt{x^2+y^2}>a\end{cases}\rightarrow g_R(r)=\begin{cases}1 & r\le a\\ 0 & r>a\end{cases}$$

Substituting $g_R(r)$ in

$$G_0(\rho,\phi)=G_0(\rho)=2\pi\int_0^\infty r g_R(r)J_0(2\pi r\rho)dr$$

$$G_0(\rho)=2\pi\int_0^a rJ_0(2\pi r\rho)dr$$

Using the the integral identity: $\int_0^u u'J_0(u')du' = uJ_1(u)$

$$r'=2\pi r\rho \qquad r=0\rightarrow r'=0 \quad \textit{and}\ r=a \quad r'=2\pi a\rho$$

$$G_0(\rho)=\frac{1}{2\pi\rho^2}\int_0^a 2\pi r\rho J_0(2\pi r\rho)d(2\pi r\rho)=\frac{1}{2\pi\rho^2}\int_0^{2\pi a\rho} r'J_0(r')dr'$$

$G_0(\rho)=\frac{1}{2\pi\rho^2}2\pi a\rho J_1(2\pi a\rho)=a\frac{J_1(2\pi a\rho)}{\rho}=2\pi a^2\frac{J_1(2\pi a\rho)}{2\pi a\rho}$ with $k_\alpha=2\pi\rho$

$G_0(k_\alpha)=F\left(k_\alpha\right)=2\pi a^2\left[\frac{J_1(k_\alpha a)}{k_\alpha a}\right]$ where J_1 is a first order Bessl function.

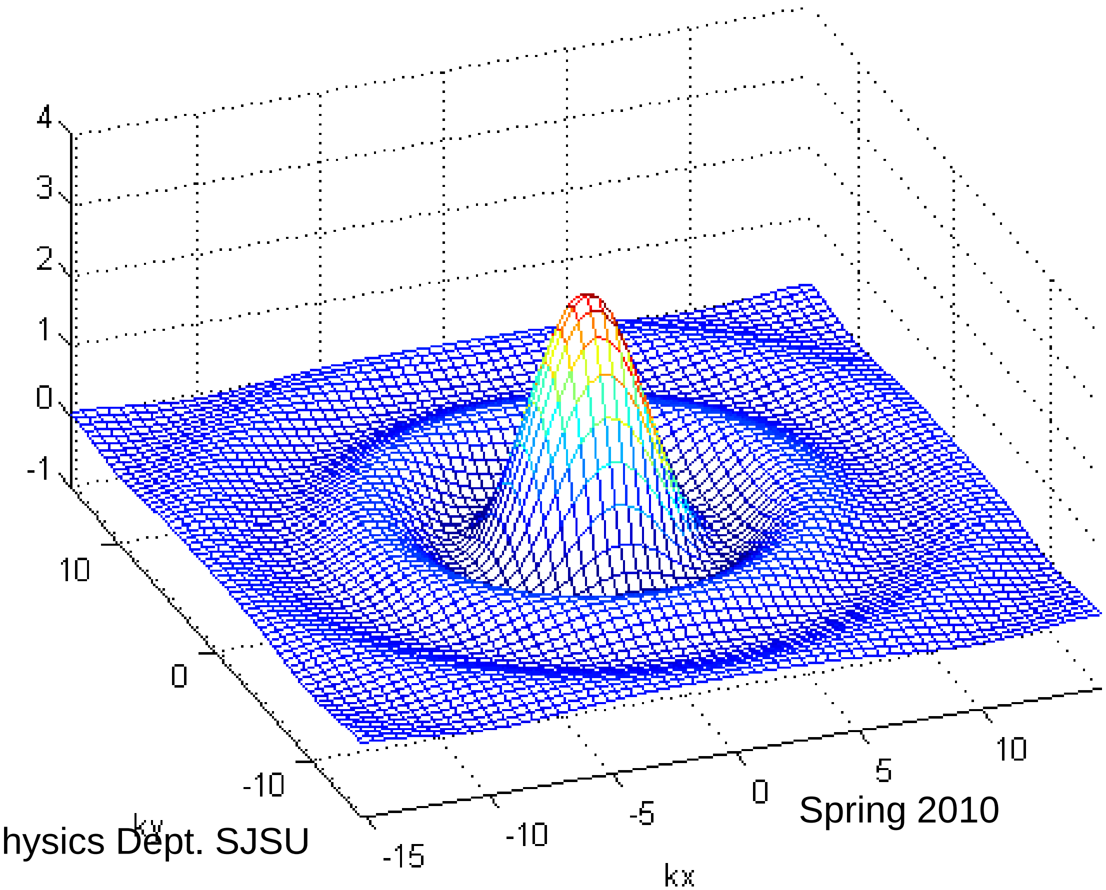
Circular aperture with Bessel functions in MATLAB

$$g(x,y)=\begin{cases}1 & \sqrt{x^2+y^2}\le a\\ 0 & \sqrt{x^2+y^2}>a\end{cases}\rightarrow g_R(r)=\begin{cases}1 & r\le a\\ 0 & r>a\end{cases}$$

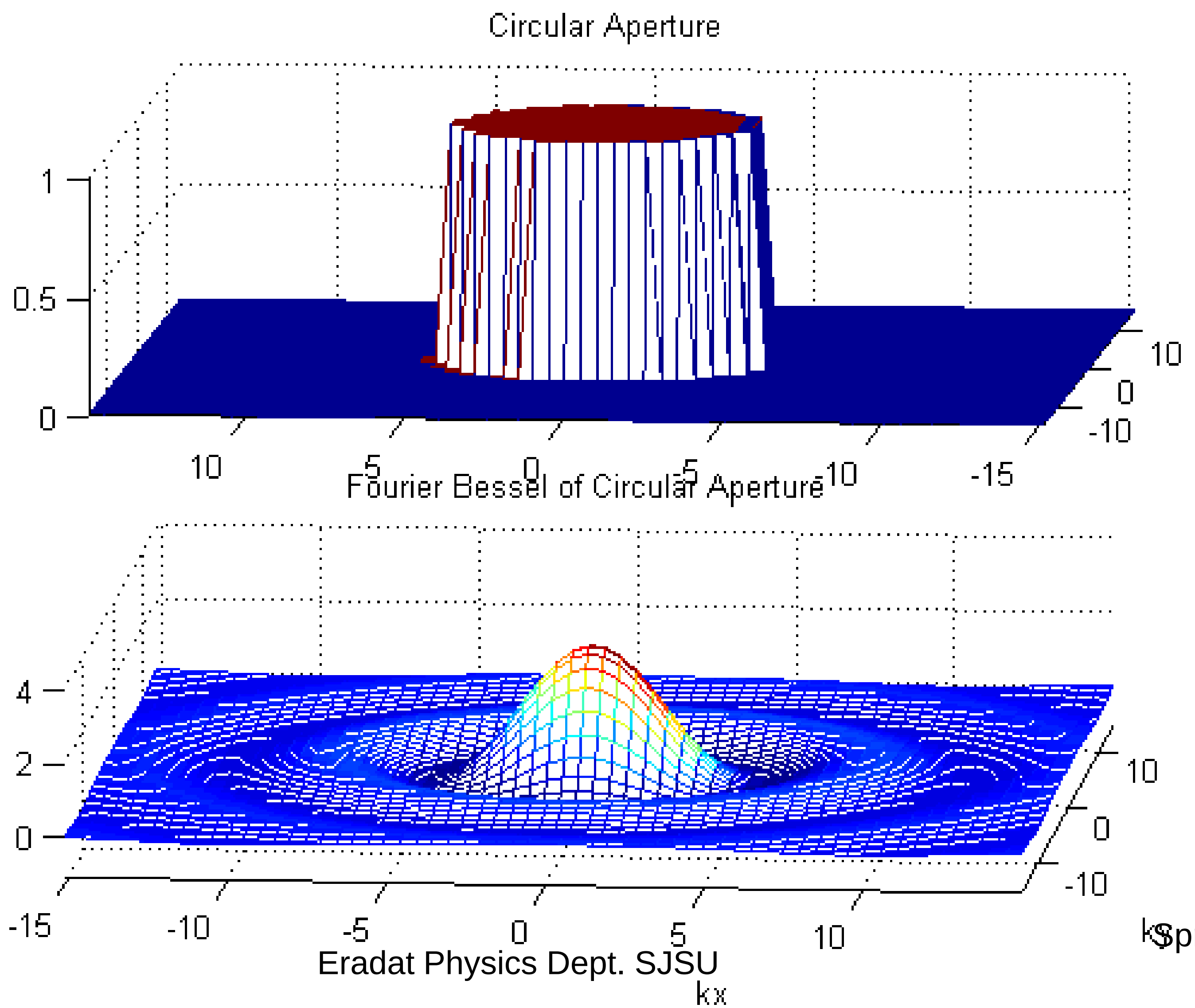
$$G_0(k_\alpha)=F(k_\alpha)=2\pi a^2\left[\frac{J_1(k_\alpha a)}{k_\alpha a}\right]$$

```
x=(-15.1:0.5:14.9);
y=(-15.1:0.5:14.9);
A=y.'*x;
i_index=0;
for i=-15.1:0.5:14.9
   j_index=0;
   i_index=i_index+1;
   for j=-15.1:0.5:14.9
      j_index=j_index+1;
      r=sqrt(i^2+j^2);
      if r <=5
         A(i_index,j_index)=1;
      else A(i_index,j_index)=0;
      end
   end
end
subplot(2,1,1);
mesh(x,y,A);
title('Circular Aperture')
axis([-15.1 14.9 -15.1 14.9 0 1]);
a=1;
kx=(-15.1:0.5:14.9);
ky=(-15.1:0.5:14.9);
[kax,kay]=meshgrid(kx,ky);

ka=sqrt(kax.^2+kay.^2);
Gka=2*pi*a^2.*besselj(1,ka)./(ka*a);
subplot(2,1,2);
mesh(kx,ky,Gka);
xlabel('kx'); ylabel('ky');
axis([-15.1 14.9 -15.1 14.9 -1 4]);
title('Fourier Bessel of Circular Aperture')
```

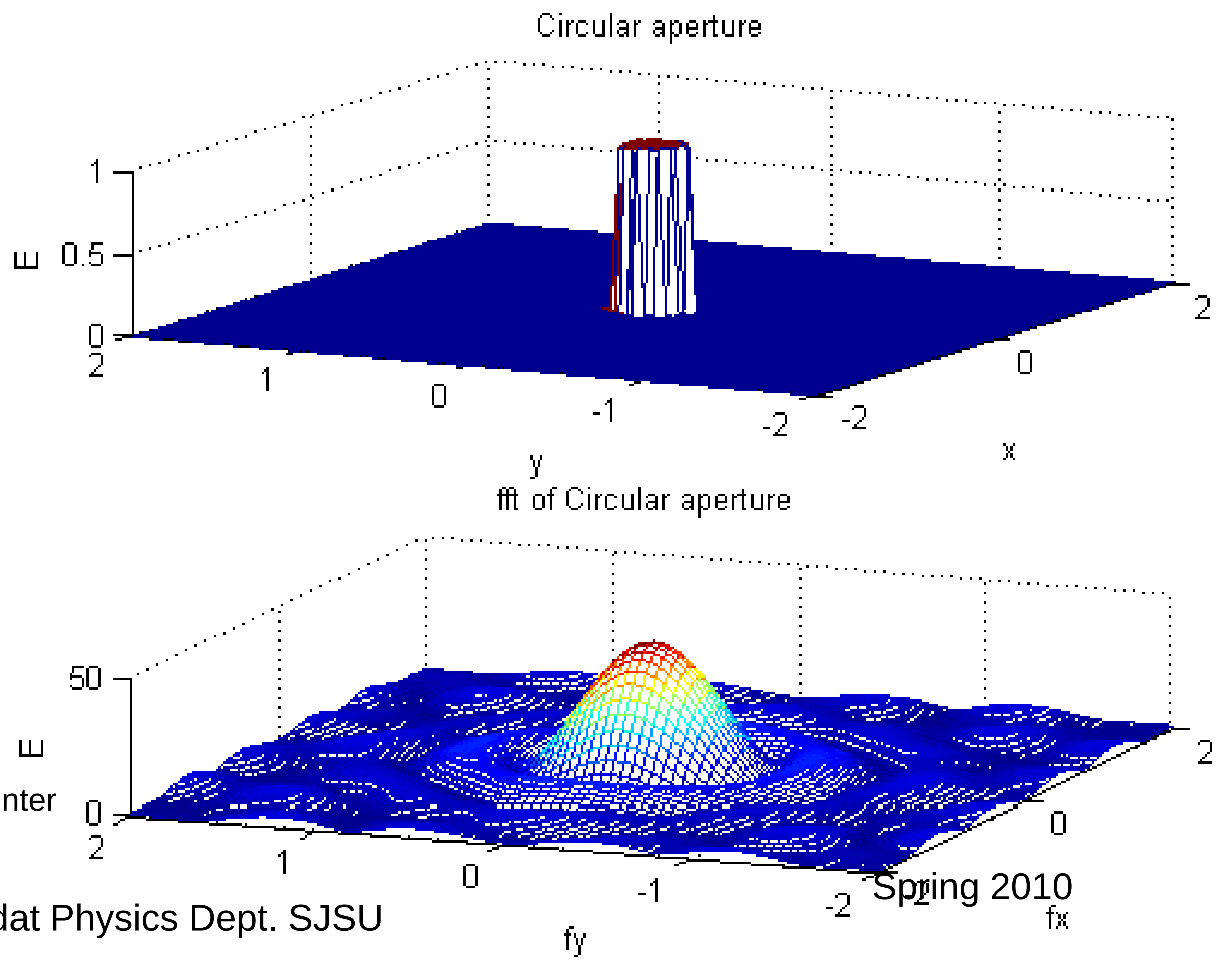


Circular aperture with Bessel functions in MATLAB



Circular aperture with FFT in MATLAB

```
%PHYS 258 spring 07, Nayer Eradat
%A program to plot a circular aperture
function
%and its Fourier transform using fft and shift
fft function
x=(-2:0.05:2);
y=(-2:0.05:2);
A=y.'*x;
i_index=0;
for i=-2:0.05:2
    j_index=0;
    i_index=i_index+1;
    for j=-2:0.05:2
        j_index=j_index+1;
        r=sqrt(i^2+j^2);
        if r <=0.2
            A(i_index,j_index)=1;
        else A(i_index,j_index)=0;
        end
    end
end
subplot(2,1,1);
mesh(x,y,A); %3D plot
xlabel('x'); ylabel('y'); zlabel('E');
title('Circular aperture');
fft_v=abs(fft2(A));
fft_val=fftshift(fft_v);
%shift zero-frequency component to center
of spectrum
subplot(2,1,2);
mesh(x,y,fft_val);
xlabel('fx'); ylabel('fy'); zlabel('E');
title('fft of Circular aperture');
```



4.4 Examples of Fraunhofer diffraction patterns

- We can apply the results of Fraunhofer approximation to calculate the complex field distribution pattern across any given aperture.
- The physically observable quantity is the intensity of the radiation rather than the field strength.
- In the following examples we will calculate the intensity distributions across the apertures.

Screen amplitude transmittance function

$$\text{Screen amplitude transmittance function} = \frac{\text{complex field amplitude imediately behind the screen}}{\text{incident complex field amplitude}}$$

Screen amplitude transmittance for an infinite opaque screen:

$$t_A(\xi,\eta) = \begin{cases} 1 & \text{in the aperture} \\ 0 & \text{outside the aperture} \end{cases}$$

It is possible to introduce for example

a) Phase mask: spatial patterns of phase shift by means of transparent plates of various thickness

b) Amplitude mask: spatial attenuation by placing an absorbing photographic transparency with real values between $0 \le t_A \le 1$

These two techniques extend all realizable values of t_A over the complex planes within the unit circle.

4.4.1 Rectangular aperture I

Goal: calculate the intensity of the Fraunhofer diffraction pattern at a distance z from a rectangular aperture located on an infinite opaque screen. Aperture amplitude transmittance:

$$t_A(\xi,\eta) = rect\left(\frac{\xi}{2w_X}\right) rect\left(\frac{\eta}{2w_Y}\right)$$

where w_X and w_Y are the half widths of the aperture in ξ and η directions.

Illumination: a unit-amplitude, normally incident, monochromatic plane wave: For such an illumination the field distribution just across the aperture is the transmittance function t_A, and the Fraunhofer diffraction pattern is:

$$U(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \underbrace{\int\int_{-\infty}^{\infty} U(\xi,\eta) e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)} d\xi d\eta}_{}$$

Fourier transform of the aperture distribution evaluated at $f_X = \frac{x}{\lambda z}$ and $f_Y = \frac{y}{\lambda z}$

$$U(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \mathcal{F}\left\{U(\xi,\eta)\right\}\Big|_{f_X = x/\lambda z,\ f_Y = y/\lambda z}$$

4.4.1 Rectangular aperture II

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\mathcal{F}\left\{t_A(\xi,\eta)\right\}\Big|_{f_X=x/\lambda z,\ f_Y=y/\lambda z}$$

$$t_A(\xi,\eta)=rect\left(\frac{\xi}{2w_X}\right)rect\left(\frac{\eta}{2w_Y}\right)$$

$$\mathcal{F}\left\{t_A(\xi,\eta)\right\}=2w_X\sin c\left(2w_X f_X\right)\ 2w_Y\sin c\left(2w_Y f_Y\right)$$ with $A=4w_X w_Y$

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}A\sin c\left(2w_X f_X\right)\sin c\left(2w_Y f_Y\right)\Big|_{f_X=x/\lambda z,\ f_Y=y/\lambda z}$$

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}A\sin c\left(\frac{2w_X x}{\lambda z}\right)\sin c\left(\frac{2w_Y y}{\lambda z}\right)$$

$$I(x,y)=|U(x,y)|^2=\frac{A^2}{\lambda^2 z^2}\sin c^2\left(\frac{2w_X x}{\lambda z}\right)\sin c^2\left(\frac{2w_Y y}{\lambda z}\right)$$

4.4.1 Rectangular aperture III

$$I(x,y) = |U(x,y)|^2 = \frac{A^2}{\lambda^2 z^2} \sin c^2\left(\frac{2w_X x}{\lambda z}\right) \sin c^2\left(\frac{2w_Y y}{\lambda z}\right)$$

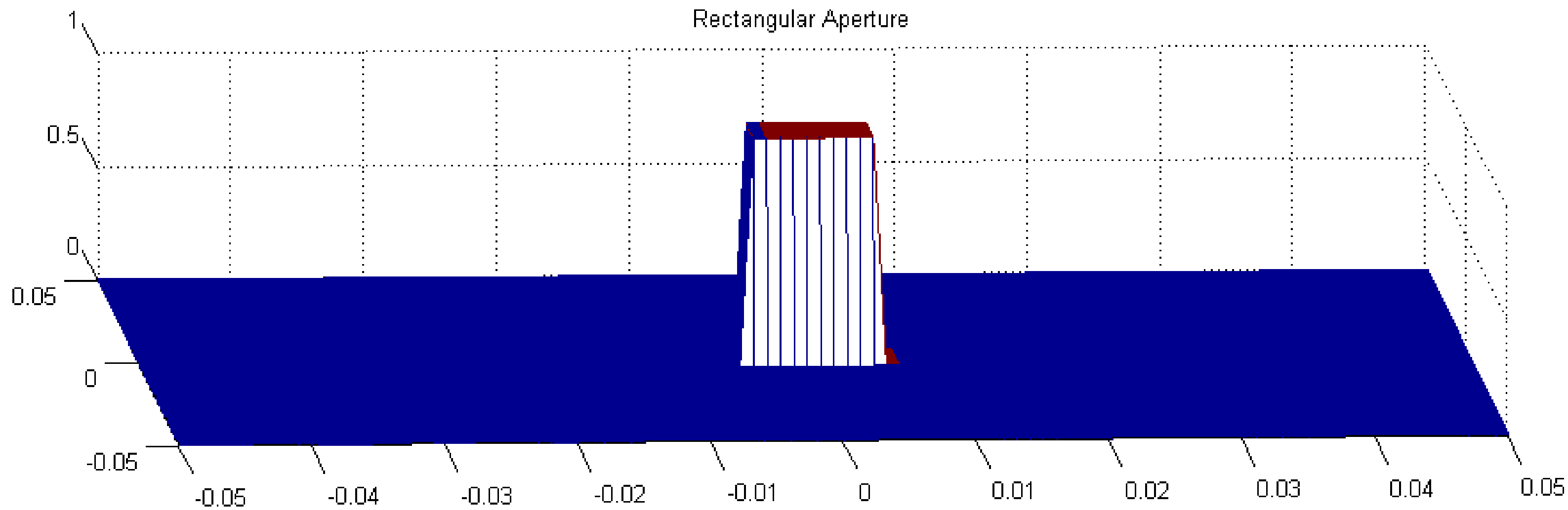
Exercise: prove that width of the maine lobe or distance between the first two zeros is $\Delta x = \frac{\lambda z}{w_X}$.

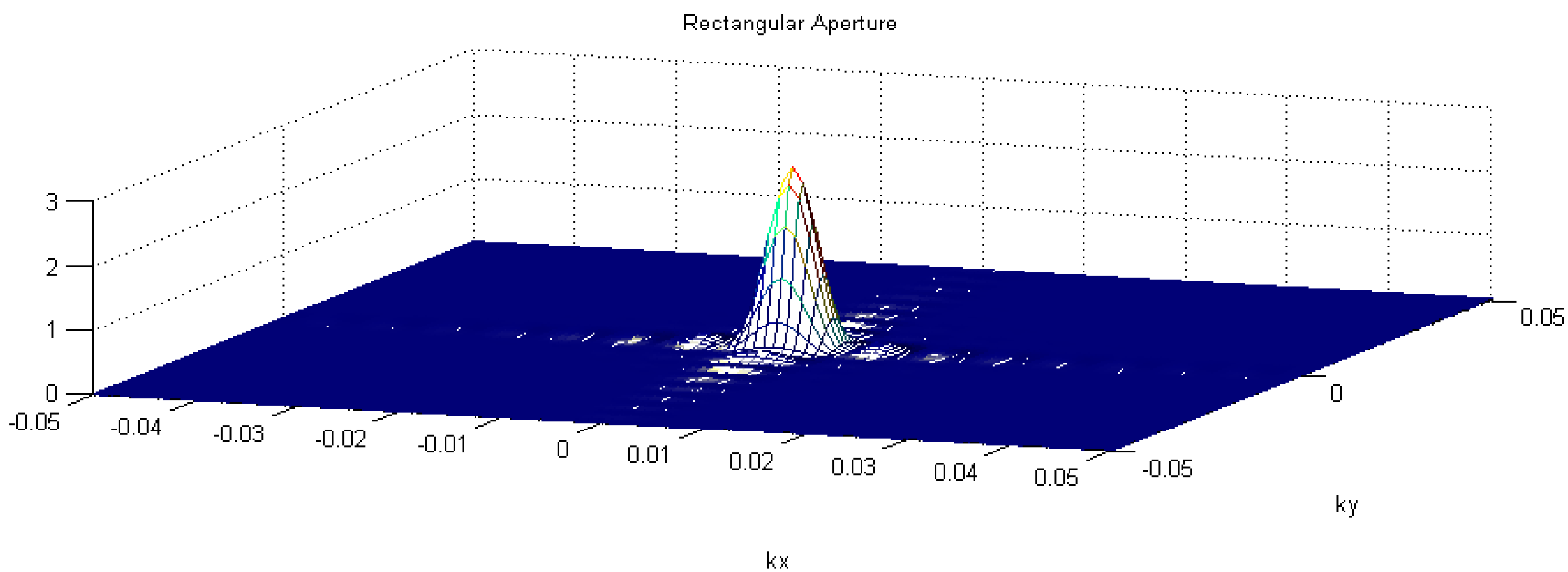
Solution: we need to find roots of the I, when y=0, we have

$\sin c^2\left(\frac{2w_Y y}{\lambda z}\right) = 1$ so we need to require

$$\sin c^2\left(\frac{2w_X x}{\lambda z}\right) = 0 \rightarrow \frac{\sin\left(\pi \frac{2w_X x}{\lambda z}\right)}{\pi \frac{2w_X x}{\lambda z}} = 0 \rightarrow \pi \frac{2w_X x}{\lambda z} = m\pi \rightarrow x = \frac{m\lambda z}{2w_X}$$

with $m = \pm 1$ we get $x_+ = \frac{\lambda z}{2w_X}$ and $x_- = -\frac{\lambda z}{2w_X} \rightarrow \Delta x = \frac{\lambda z}{w_X}$


Rectangular Aperture
1
0.5
0
0.05
0
-0.05
-0.05
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
0.04
0.05


Rectangular Aperture
3
2
1
0
-0.05
-0.04
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.05
0
-0.05
ky
kx

4.4.2 Circular aperture I

Goal: calculate the intensity of the Fraunhofer diffraction pattern at a distance z from a circular aperture of radius q located on an infinite opaque screen. Aperture amplitude transmittance:

$$t_A(q) = circ\left(\frac{q}{w}\right)$$

Circular symmetry suggests using the cylinderical coordinates and the Fourier-Bessel transformation. The Fraunhofer diffraction pattern is:

$$U(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \underbrace{\int\int_{-\infty}^{\infty} U(\xi,\eta) e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)} d\xi d\eta}$$

Fourier-Bessel transform of the aperture distribution evaluated at $f_X = \frac{x}{\lambda z}$ and $f_Y = \frac{y}{\lambda z}$

$$U(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}r^2} \mathcal{B}\left\{U(q)\right\}\Big|_{\rho=r/\lambda z}$$

where $r = \sqrt{x^2+y^2}$ is the radius in the aperture plane and $\rho = \sqrt{f_X^2 + f_Y^2}$ is the radius in the spatial frequency plane.

4.4.2 Circular aperture II

Illumination: a unit-amplitude, normally incident, monochromatic plane wave:
For such an illumination the field distribution just across the aperture is the transmittance function t_A,

$$U(r) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}r^2} \mathcal{B}\left\{t_A(q)\right\}\Big|_{\rho=r/\lambda z} = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}r^2} \mathcal{B}\left\{circ\left(\frac{q}{w}\right)\right\}\Bigg|_{\rho=r/\lambda z}$$

$$\mathcal{B}\left\{circ\left(\frac{q}{w}\right)\right\} = A\frac{J_1(2\pi w\rho)}{\pi w\rho}$$ where $A = \pi w^2$. With $\rho = \frac{r}{\lambda z}$; $2\pi w\rho = \frac{kwr}{z}$

$$U(r) = \frac{A}{j\lambda z} e^{jkz} e^{j\frac{k}{2z}r^2}\left[2\frac{J_1(kwr/z)}{kwr/z}\right]$$

$$I(r) = \left(\frac{A}{\lambda z}\right)^2\left[2\frac{J_1(kwr/z)}{kwr/z}\right]^2$$ $\leftarrow$ The Airy pattern.

Width of the central lobe measured along the x and y axis: $d = 1.22\frac{\lambda z}{w}$

4.4.2 Circular aperture III

Exercise: Prove that width of the central lobe measured along the x and y axis on the Airy pattern is: $d = 1.22\dfrac{\lambda z}{w}$

we start from the Airy pattern: $I(r) = \left(\dfrac{A}{\lambda z}\right)^2 \left[2\dfrac{J_1(kwr/z)}{kwr/z}\right]^2 = 0$ for the roots*.

$$\frac{J_1(kwr/z)}{kwr/z} = 0 \text{ for r} \neq 0 \quad \text{so} \quad \frac{kwr}{z} = \frac{2\pi wr}{\lambda z} = 3.8317 \rightarrow r = \frac{3.8317}{3.14}\frac{\lambda z}{w}$$

$$r = 1.2203\frac{\lambda z}{w}$$

Using the table we an calculate the other zeros

*First few roots of the Bessel functions for the first kind using BesselZeros[n,k}
in Mathematica

zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

The grating equation

Condition for the constructive interference for a light passing through a transmission grating:

$$n_2 \Lambda \sin\theta_2 - n_1 \Lambda \sin\theta_1 = m\lambda$$

The grating equation

$$n_2 \sin\theta_2 = n_1 \sin\theta_1 + m\lambda \frac{\lambda}{\Lambda}$$

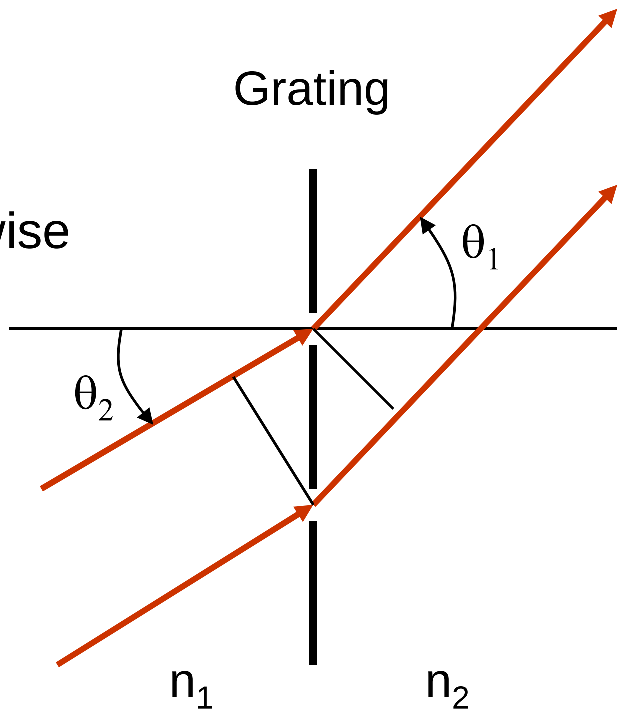
A "positive" diffraction order (m>0) $\longleftrightarrow \theta_2 > \theta_1$

A "nagative" diffraction order (m<0) $\longleftrightarrow \theta_2 < \theta_1$

θ_2 and θ_1 are signed angles poitive counterclocckwise

$\theta_2 > \theta_1$ corresponds to the zeroth order

For a reflection grating bothe incident and reflected rays are on the same side so $n_1 = n_2 = n$



4.4.3 Thin sinusoidal amplitude grating I

Goal: calculate the intensity of the Fraunhofer diffraction pattern at a distance z from a thin sinusoidal amplitude grating.

The amplitude trnsmittance function is:

$$t_A(\xi,\eta) = \left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right] rect\left(\frac{\xi}{2w}\right) rect\left(\frac{\eta}{2w}\right)$$

We have assumed that the grating structure is bounded by a square aperture of width $2w$.

m is the peak-to-peak change of amplitude transmittance across the screen.

f_0 is the sptial freuency of the grating.

thin means the structure can be modeled by a simple amplitude transmittance (no effect on the phase).

Illumination: a unit-amplitude plane wave

t_A: the field distribution across the aperture.

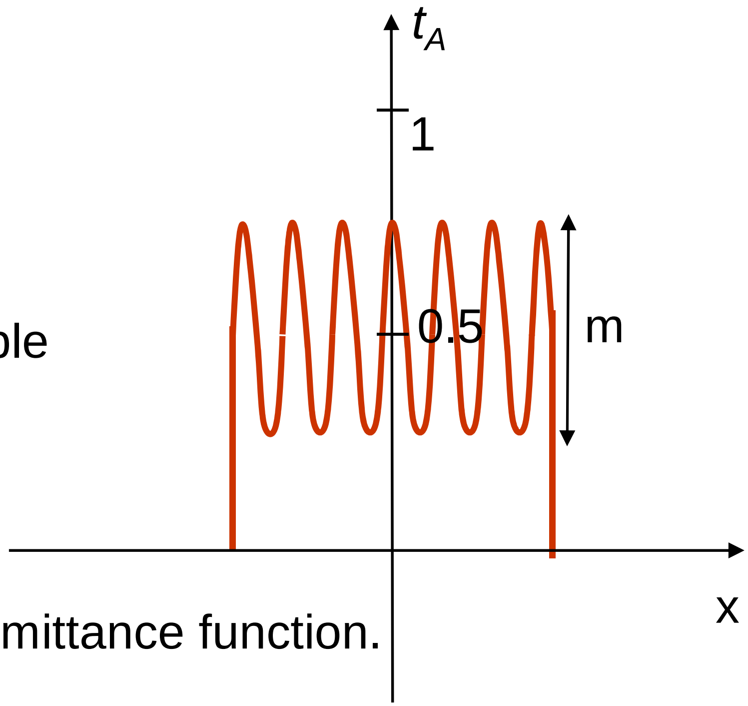


Figure: cross section of the grating amplitude transmittance function.

4.4.3 Thin sinusoidal amplitude grating II

The Fraunhofer diffraction pattern is the Fourier transfor of t_A:

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}\left(x^2+y^2\right)}\mathcal{F}\left\{U(\xi,\eta)\right\}\Big|_{f_X,f_Y} = \frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}\left(x^2+y^2\right)}\mathcal{F}\left\{t_A(\xi,\eta)\right\}\Big|_{f_X,f_Y}$$

but first:

$$\mathcal{F}\left\{\frac{1}{2}+\frac{m}{2}\cos(2\pi f_0\xi)\right\}=\frac{1}{2}\delta\left(f_X,f_Y\right)+\frac{m}{4}\delta\left(f_X+f_0,f_Y\right)+\frac{m}{4}\delta\left(f_X-f_0,f_Y\right)$$

$$\mathcal{F}\left\{rect\left(\frac{\xi}{2w}\right)rect\left(\frac{\eta}{2w}\right)\right\}=Asnic\left(2wf_X\right)snic\left(2wf_Y\right)$$ where $A=4w^2$ is the area of aperture bounding the grating.

$$\mathcal{F}\left\{t_A(\xi,\eta)\right\}=\frac{A}{2}snic\left(2wf_Y\right)\left\{snic\left(2wf_X\right)+\frac{m}{2}snic(2w(f_X+f_0)+\frac{m}{2}snic(2w(f_X-f_0)\right\}$$

With $f_X=x/\lambda z$ and $f_Y=y/\lambda z$

$$U(x,y)=\frac{A}{j\lambda z}e^{jkz}e^{j\frac{k}{2z}\left(x^2+y^2\right)}snic\left(\frac{2wy}{\lambda z}\right)\left\{snic\left(\frac{2wx}{\lambda z}\right)+\frac{m}{2}\sin c(\frac{2w}{\lambda z}(f_X+f\ \lambda z)+\frac{m}{2}\sin c(\frac{2w}{\lambda z}(f_X-f_0\lambda z)\right\}$$

4.4.3 Thin sinusoidal amplitude grating III

And finally

$$I(x,y)=\left[\frac{A}{j\lambda z}\right]^2\left[e^{jkz}e^{j\frac{k}{2z}(x^2+y^2)}\right]snic^2\left(\frac{2wy}{\lambda z}\right)$$

$$\left\{snic\left(\frac{2wx}{\lambda z}\right)+\frac{m}{2}\sin c(\frac{2w}{\lambda z}(f_X+f_0\lambda z)+\frac{m}{2}\sin c(\frac{2w}{\lambda z}(f_X-f_0\lambda z)\right\}^2$$

$$I(x,y)=\left[\frac{A}{j\lambda z}\right]^2 snic^2\left(\frac{2wy}{\lambda z}\right)\left\{snic\left(\frac{2wx}{\lambda z}\right)+\frac{m}{2}\sin c(\frac{2w}{\lambda z}(f_X+f_0\lambda z)+\frac{m}{2}\sin c(\frac{2w}{\lambda z}(f_X-f_0\lambda z)\right\}^2$$

For $f_0 \gg 1/w$ or for a very fine grating rulling the overlap between the sinc functions is negligible and I is approximately eual to the sum of squared amplitudes.

$$I(x,y)\approx\left[\frac{A}{j\lambda z}\right]^2 snic^2\left(\frac{2wy}{\lambda z}\right)\left\{snic^2\left(\frac{2wx}{\lambda z}\right)+\frac{m^2}{4}\sin c^2(\frac{2w}{\lambda z}(f_X+f_0\lambda z)+\frac{m^2}{4}\sin c^2(\frac{2w}{\lambda z}(f_X-f_0\lambda z)\right\}$$

$\eta =$ Diffraction efficiency $=$ fraction of the power in a single order of the Fraunhofer diff. patteren.

It can be found from: $\mathcal{F}\left\{\frac{1}{2}+\frac{m}{2}\cos(2\pi f_0\xi)\right\}=\frac{1}{2}\delta(f_X,f_Y)+\frac{m}{4}\delta(f_X+f_0,f_Y)+\frac{m}{4}\delta(f_X-f_0,f_Y)$

Since the delta functions determine power of the pattern and sinc functions only spread them.

$\eta_0=\left(\frac{1}{2}\right)^2=0.25,\quad \eta_1=\left(\frac{m}{4}\right)^2=\frac{m^2}{16},\quad \eta_{-1}=\left(\frac{m}{4}\right)^2=\frac{m^2}{16}$ so $\eta_{1\,\max}=\frac{1}{16}=6.25\%$ and total power in 3 orders is 3/8. The rest of the power is lost by absorption of the grating.

4.4.4 Thin sinusoidal phase grating I

Goal: calculate the intensity of the Fraunhofer diffraction pattern at a distance z from a thin sinusoidal phase grating.

The amplitude trnsmittance function is:

$$t_A(\xi,\eta) = \underbrace{e^{\left[j\frac{m}{2}\sin(2\pi f_0 \xi)\right]}}_{\text{Sinusoidal phase difference introduced by the grating}} rect\left(\frac{\xi}{2w}\right) rect\left(\frac{\eta}{2w}\right)$$

average phase delay caused by grating is elliminated by proper choice of teference.

We have assumed that the grating structure is bounded by a square aperture of width $2w$.

m is the peak-to-peak excursion of the phase delay.

f_0 is the sptial freuency of the grating.

thin means the structure can be modeled by a simple phase transmittance (no effect on amplitude).

Illumination: a unit-amplitude plane wave

t_A: the field distribution across the aperture.

4.4.4 Thin sinusoidal phase grating II

$$t_A(\xi,\eta) = e^{\left[j\frac{m}{2}\sin(2\pi f_0\xi)\right]} rect\left(\frac{\xi}{2w}\right) rect\left(\frac{\eta}{2w}\right)$$

$$U(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}\left(x^2+y^2\right)} \mathcal{F}\left\{U(\xi,\eta)\right\}\Big|_{f_X,f_Y} = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}\left(x^2+y^2\right)} \mathcal{F}\left\{t_A(\xi,\eta)\right\}\Big|_{f_X,f_Y}$$

Using the Bessel function identity: $e^{\left[j\frac{m}{2}\sin(2\pi f_0\xi)\right]} = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) e^{j2\pi q f_0 \xi}$

$$\mathcal{F}\left\{t_A(\xi,\eta)\right\} = \mathcal{F}\left\{\sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) e^{j2\pi q f_0\xi}\right\} \otimes \mathcal{F}\left\{rect\left(\frac{\xi}{2w}\right) rect\left(\frac{\eta}{2w}\right)\right\}$$

$$\mathcal{F}\left\{t_A(\xi,\eta)\right\} = \left[\sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\delta(f_X - qf_0, f_Y)\right] \otimes \left[A\sin c(2wf_X)\sin c(2wf_Y)\right]$$

$$\mathcal{F}\left\{t_A(\xi,\eta)\right\} = \sum_{q=-\infty}^{\infty} AJ_q\left(\frac{m}{2}\right)\sin c\left[2w\left(f_X - qf_0\right)\right]\sin c\left(2wf_Y\right)$$

$$U(x,y) = \frac{A}{j\lambda z} e^{jkz} e^{j\frac{k}{2z}\left(x^2+y^2\right)} \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\sin c\left[\frac{2w}{\lambda z}\left(x - qf_0\lambda z\right)\right]\sin c\left(\frac{2wy}{\lambda z}\right)$$

4.4.4 Thin sinusoidal phase grating III

$$t_A(\xi,\eta) = e^{\left[j\frac{m}{2}\sin(2\pi f_0\xi)\right]} rect\left(\frac{\xi}{2w}\right) rect\left(\frac{\eta}{2w}\right)$$

$$U(x,y) = \frac{A}{j\lambda z} e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)} \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) \sin c\left[\frac{2w}{\lambda z}(x - qf_0\lambda z)\right] \sin c\left(\frac{2wy}{\lambda z}\right)$$

$$I = \left(\frac{A}{j\lambda z} e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)} \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right) \sin c\left[\frac{2w}{\lambda z}(x - qf_0\lambda z)\right] \sin c\left(\frac{2wy}{\lambda z}\right)\right)^2$$

For $f_0 \gg 1/w$ or for a very fine grating rulling the overlap between the sinc functions is negligible and I is approximately eual to the sum of squared amplitudes.

$$I \approx \left(\frac{A}{\lambda z}\right)^2 \sum_{q=-\infty}^{\infty} J_q^2\left(\frac{m}{2}\right) \sin c^2\left[\frac{2w}{\lambda z}\left(x - \underbrace{qf_0\lambda z}_{\substack{\text{Displacement}\\\text{of the order}\\\text{fromthe center}}}\right)\right] \sin c^2\left(\frac{2wy}{\lambda z}\right)$$

We see that introduction of the sinusoidal phase grating has deflected power from the zeroth order to the higher ordes.

Peak intensity of the qth order $= \left[\frac{A}{\lambda z} J_q\left(\frac{m}{2}\right)\right]^2$

It happens when $y = 0$ and $x - qf_0\lambda z = 0 \rightarrow x = qf_0\lambda z$

4.4.4 Thin sinusoidal phase grating IIII

Displacement of qth order from the center of the pattern $= qf_0\lambda z$

For $q = 0$ or $zero$th order $y = 0$ and $x = 0$

For $q = \pm 1$ or first order $y = 0$ and $x = \pm f_0\lambda z$ function of frequency of the grating lining, wavelength, and distance of observation.

So for spectroscopy in the blue region we need hight f_0 grating or larger spectrometer.

$\eta =$ Diffraction efficiency $=$ fraction of the power in a single order of the Fraunhofer diff. patteren.

It can be found from: $\mathcal{F}\left\{t_A(\xi,\eta)\right\} = \left[\sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\delta(f_X - qf_0, f_Y)\right] \otimes \left[A\sin c(2wf_X)\sin c(2wf_Y)\right]$

Since the delta functions determine power of the pattern and sinc functions only spread them.

$\eta_0 = J_q^2\left(\frac{m}{2}\right)$

Plot $\eta_0 = J_q^2\left(\frac{m}{2}\right)$ we see that when m/2 is root of J_0 then the central lobe wanishes. $\eta_{1\ max} = 33.8\%$

which is much greater than that of the sinusoidal amplitude grating which is $\frac{1}{16} = 6.25.$

There is no power absorption and sum of the power in all orders is equal to the total incident power.

The distribution of power between the lobes varies as m changes.

4.5 Examples of Fresnel diffraction calculations

- Based on the example we will choose a different approach to the Fresnel diffraction examples.
  - convolution representation.
  - Frequency domain approach

4.5.1 Fresnel diffraction by a square aperture I

Goal: calculate the intensity of the Fresnel diffraction pattern at a distance z from a square aperture of width $2w$ located on an infinite opaque screen. The amplitude transmittance function:

$$t_A(\xi,\eta) = rect\left(\frac{\xi}{2w}\right)rect\left(\frac{\eta}{2w}\right)$$

The complex field imediately behind the aperture:

$$U(\xi,\eta)\Big|_{z=0} = rect\left(\frac{\xi}{2w}\right)rect\left(\frac{\eta}{2w}\right)$$

Illumination: a unit-amplitude, normally incident, monochromatic plane wave

Using the convolution form of the Fresnel diffraction formula:

$$U\left(x,y\right) = \frac{e^{jkz}}{j\lambda z}\int\int_{-w}^{w} e^{j\frac{\pi}{\lambda z}\left[\left(x-\xi\right)^2-\left(y-\eta\right)^2\right]}d\xi d\eta$$

$$U\left(x,y\right) = \frac{e^{jkz}}{j}\mathcal{I}(x)\mathcal{I}(y)$$ where

$$\mathcal{I}(x) = \frac{1}{\sqrt{\lambda z}}\int_{-w}^{w} e^{j\frac{\pi}{\lambda z}\left[\left(x-\xi\right)^2\right]}d\xi d\eta \quad \text{and} \quad \mathcal{I}(y) = \frac{1}{\sqrt{\lambda z}}\int_{-w}^{w} e^{j\frac{\pi}{\lambda z}\left[\left(y-\eta\right)^2\right]}d\xi d\eta$$

4.5.1 Fresnel diffraction by a square aperture II

Change of variables:

$$\alpha = \sqrt{\frac{2}{\lambda z}}\left(\xi - x\right) \quad \text{and} \quad \beta = \sqrt{\frac{2}{\lambda z}}\left(\eta - y\right)$$

$$\mathcal{I}(x) = \frac{1}{\sqrt{2}}\int_{-\alpha_1}^{\alpha_2} e^{j\frac{\pi}{2}\alpha^2}\,d\alpha \quad \text{and} \quad \mathcal{I}(y) = \frac{1}{\sqrt{2}}\int_{-\beta_1}^{\beta_2} e^{j\frac{\pi}{2}\beta^2}\,d\beta$$

$$\alpha_1 = \sqrt{\frac{2}{\lambda z}}\left(w + x\right) \quad \text{and} \quad \alpha_2 = \sqrt{\frac{2}{\lambda z}}\left(w - x\right)$$

$$\beta_1 = \sqrt{\frac{2}{\lambda z}}\left(w + y\right) \quad \text{and} \quad \beta_2 = \sqrt{\frac{2}{\lambda z}}\left(w - y\right)$$

With the Fresnel number: $N_F = w^2 / \lambda z$ and normalized distance variables in the observation region we have:

$$X = \frac{x}{\sqrt{\lambda z}} \quad \text{and} \quad Y = \frac{y}{\sqrt{\lambda z}}$$ the limits of the integrals become:

$$\alpha_1 = \sqrt{2}\left(\sqrt{N_F} + X\right) \quad \text{and} \quad \alpha_2 = \sqrt{2}\left(\sqrt{N_F} - X\right)$$

$$\beta_1 = \sqrt{2}\left(\sqrt{N_F} + Y\right) \quad \text{and} \quad \beta_2 = \sqrt{2}\left(\sqrt{N_F} - Y\right)$$

4.5.1 Fresnel diffraction by a square aperture III

Using $\int_{\alpha_1}^{\alpha_2} e^{j\frac{\pi}{2}\alpha^2} d\alpha = \int_0^{\alpha_2} e^{j\frac{\pi}{2}\alpha^2} d\alpha - \int_0^{\alpha_1} e^{j\frac{\pi}{2}\alpha^2} d\alpha$ and the Fresnel integrals:

$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt$ and $S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$ we write

$$\mathcal{I}(x) = \frac{1}{\sqrt{2}}\left\{\left[C(\alpha_2) - C(\alpha_1)\right] + j\left[S(\alpha_2) - S(\alpha_1)\right]\right\}$$ and

$$\mathcal{I}(y) = \frac{1}{\sqrt{2}}\left\{\left[C(\beta_2) - C(\beta_1)\right] + j\left[S(\beta_2) - S(\beta_1)\right]\right\}$$

$$U(x,y) = \frac{e^{jkz}}{2j}\left\{\left[C(\alpha_2) - C(\alpha_1)\right] + j\left[S(\alpha_2) - S(\alpha_1)\right]\right\}$$
$$\times\left\{\left[C(\beta_2) - C(\beta_1)\right] + j\left[S(\beta_2) - S(\beta_1)\right]\right\}$$

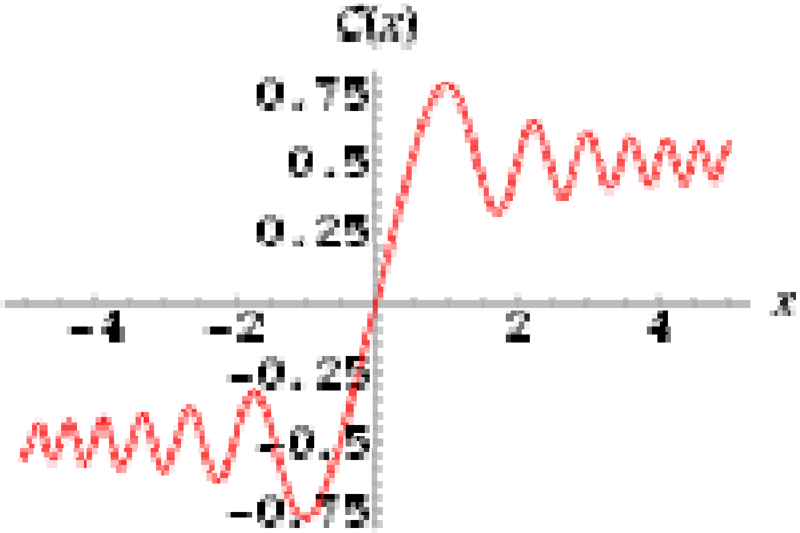
$$I(x,y) = \frac{1}{4}\left\{\left[C(\alpha_2) - C(\alpha_1)\right]^2 + \left[S(\alpha_2) - S(\alpha_1)\right]^2\right\}$$
$$\times\left\{\left[C(\beta_2) - C(\beta_1)\right]^2 + \left[S(\beta_2) - S(\beta_1)\right]^2\right\}$$

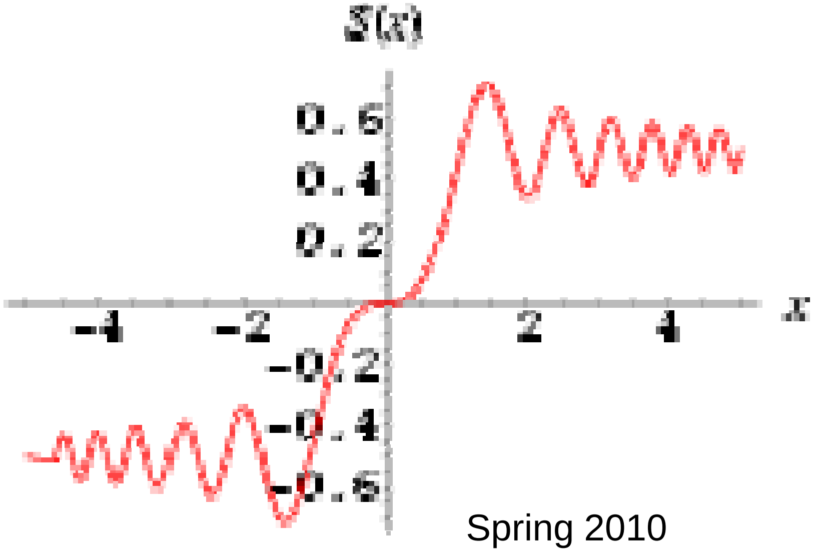
Fresnel integrals

Fresnel integrals are defined as:

$$C(z)+iS(z)=\int_0^z e^{j\frac{\pi}{2}t^2}\,dt$$

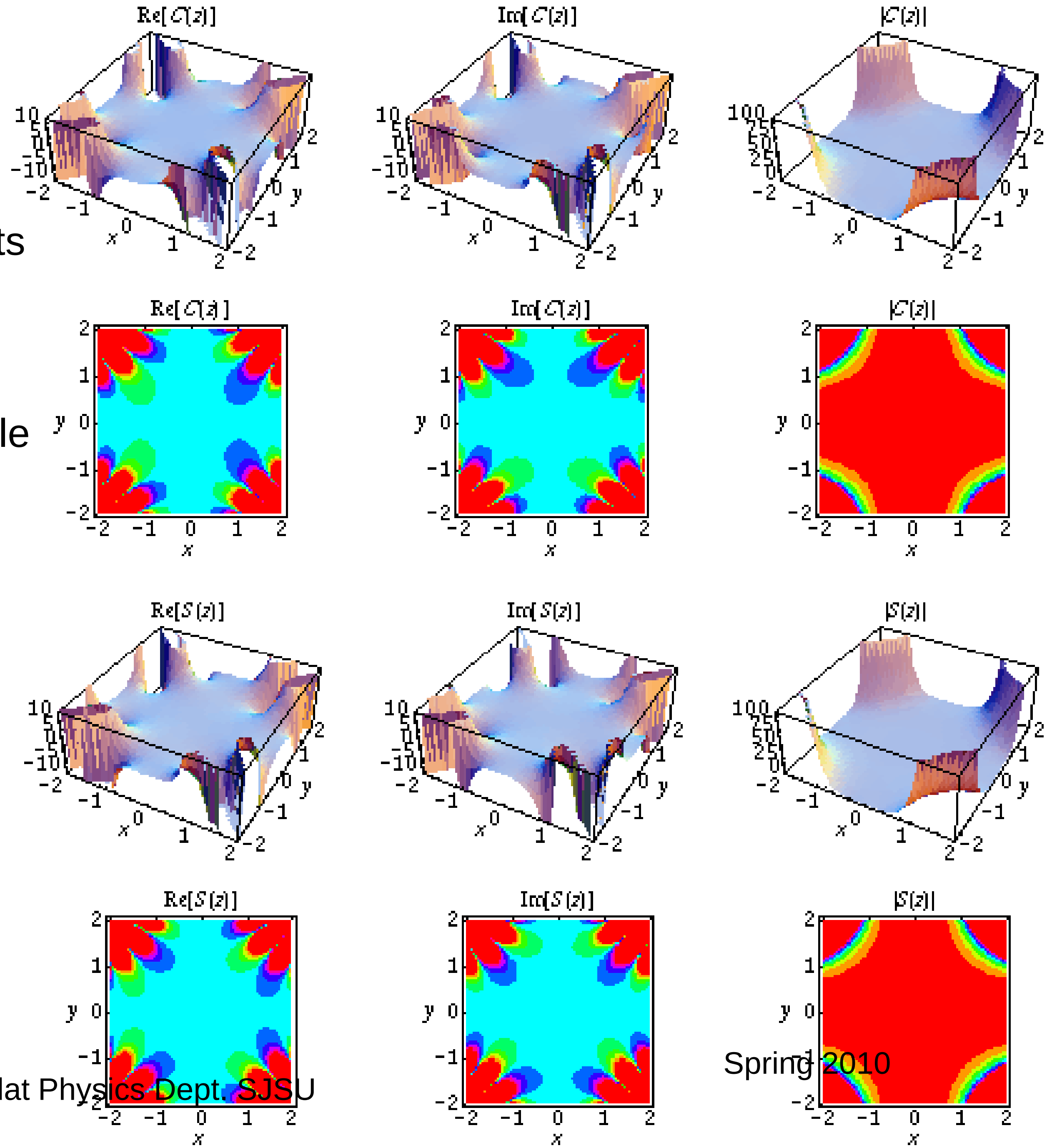
$$C(z)=\int_0^z \cos\left(\frac{\pi t^2}{2}\right)dt \quad \text{and} \quad S(z)=\int_0^z \sin\left(\frac{\pi t^2}{2}\right)dt \text{ we write}$$





The Fresnel integrals S(u) and C(u) are entire functions or integral functions i.e. they are analytical at all finite points of the complex plane.

The Fresnel integrals are tabulated and are available in many mathematical computer programs



4.5.1 Fresnel diffraction by a square aperture III

Fresnel integrals:

$$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt \; ; \; S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$$

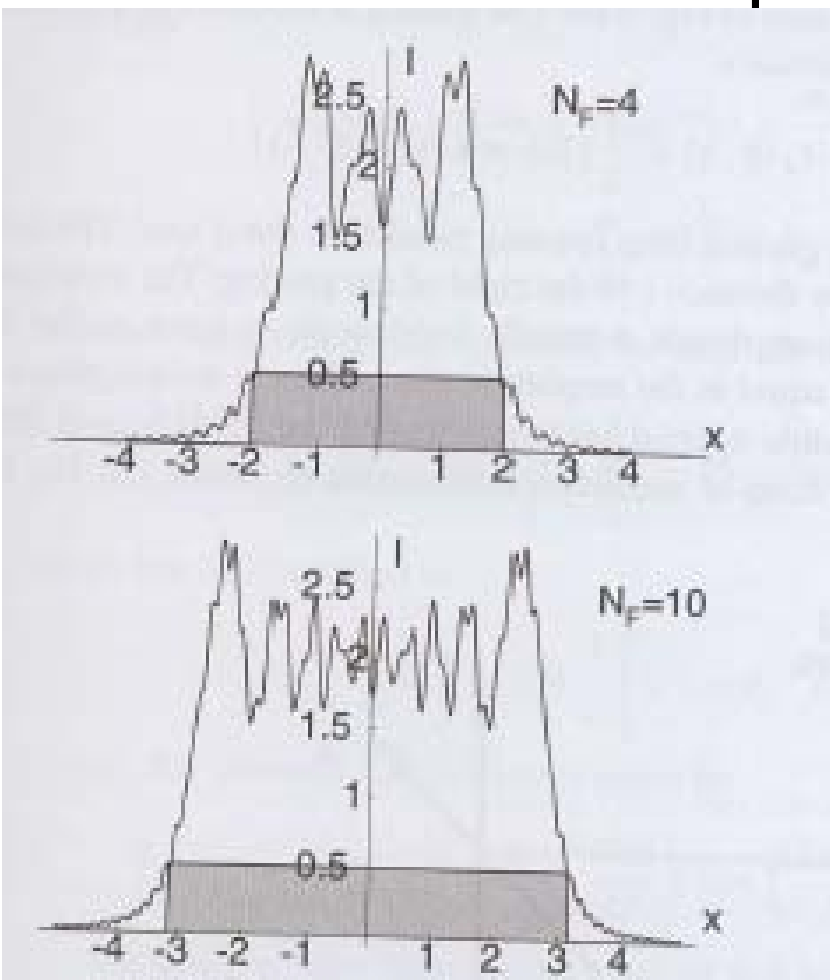
$$I(x,y) = \frac{1}{4}\left\{\left[C(\alpha_2) - C(\alpha_1)\right]^2 + \left[S(\alpha_2) - S(\alpha_1)\right]^2\right\} \times \left\{\left[C(\beta_2) - C(\beta_1)\right]^2 + \left[S(\beta_2) - S(\beta_1)\right]^2\right\}$$

$N_F = w^2 / \lambda z$ for a fixed w and $\lambda,$ as z increases, the Fresnel number decreases and the true physical distance distance represented on the $x = X\sqrt{\lambda z}$ and $y = Y\sqrt{\lambda z}$ axis are increased.

Figure shows the normalized intensity distribution along the x axis $(y = 0)$ for various normalized distances from the aperture as represented by fresnel number. As $z \to 0,\; \alpha \to \infty$ and $\beta \to \infty,\; N_F$ becomes very large and $U(x,y)$ approaches the product of a delta function and e^{jkz} and shape of the doffraction pattern approachs the shape of the aperture. Limit of the process is the geometrical optics prediction of the complex field:

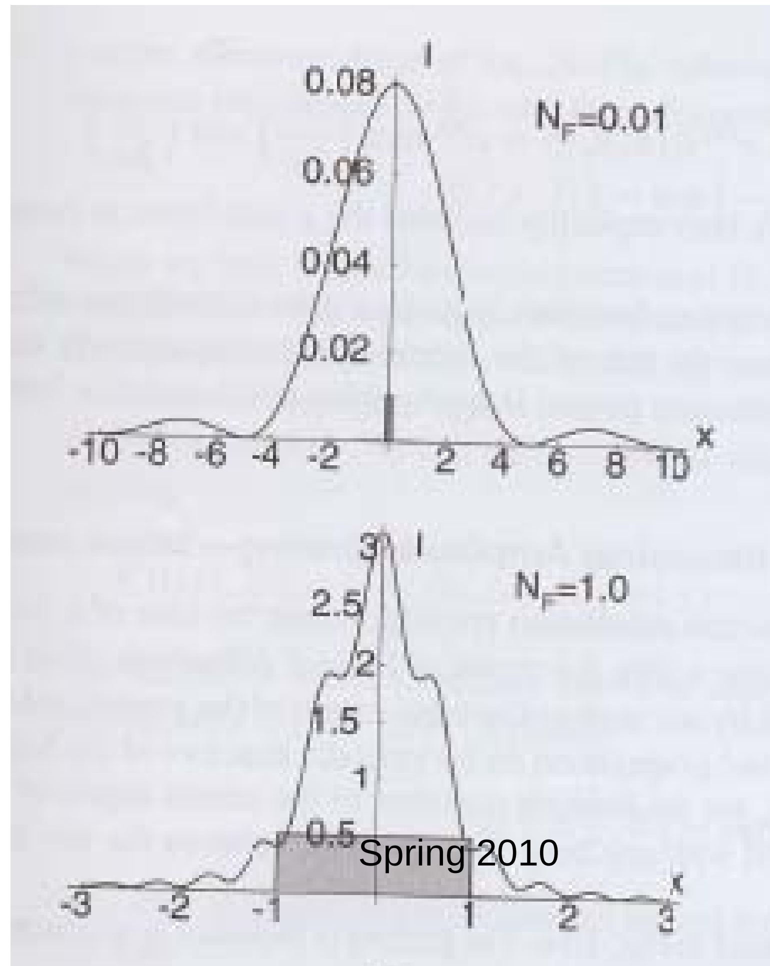
$$U(x,y) = e^{jkz} U(x,y,0) = e^{jkz}\, rect\left(\frac{x}{2w}\right) rect\left(\frac{y}{2w}\right)$$

Fresnel diffraction patterns at different distances from a square aperture.



As N_F approaches infinity diffraction pattern becomes sharp and narrow approaching the geometrical shadow of the aperture

As N_F->0 diffraction pattern becomes wide and smooth approaching Fraunhofer diffraction

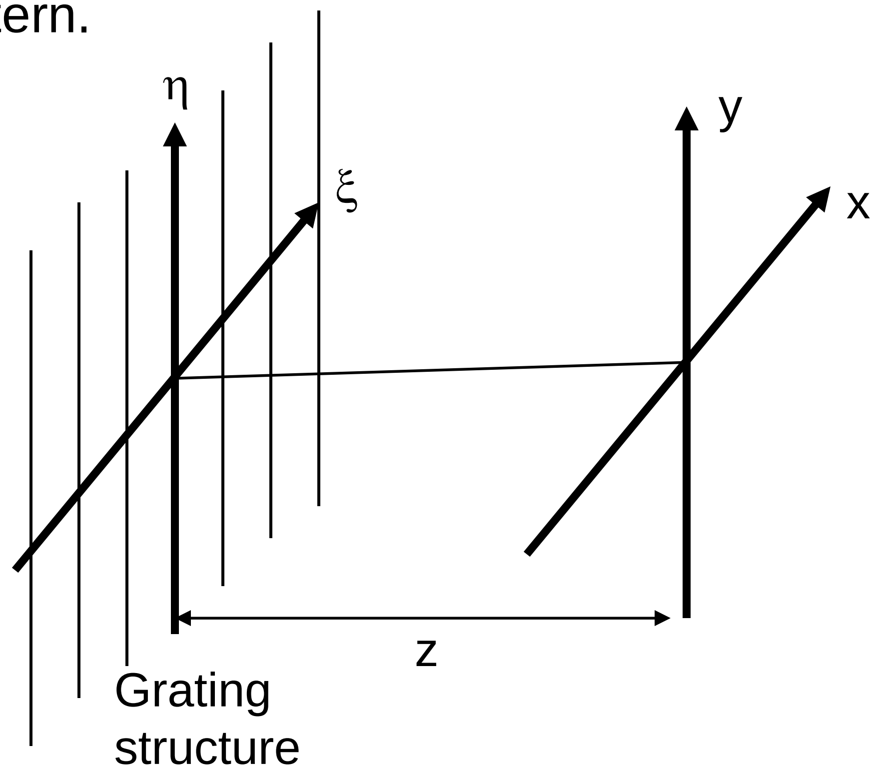


4.5 2 Fresnel diffraction by a thin sinusoidal amplitude grating-Talbot images I

Goal: calculated the Intensity distrubution of the diffraction by a thin sinusoidal amplitude grating using the Fresnel diffraction formulation.

For simplicity we neglect the finite extent of the grating.

The field transmotted by the grating has a periodic nature or we limit the attention to the central region of the pattern.



The amplitude transmittance function is modeled as:

$$t_A(\xi,\eta) = \frac{1}{2}\left[1 + m\cos(2\pi\xi / L\right]$$

Where L is the period of the lines parallel to the axis η.

Illumination: a unit amplitude normally incident plane wave.

The field immediately behind the grating is t_A.

4.5 2 Fresnel diffraction by a thin sinusoidal amplitude grating-Talbot images I

We will use the convolution form of the Fresnel diffraction equation

$$U\left(x,y\right)=\frac{e^{jkz}}{j\lambda z}\int\int_{-\infty}^{\infty}U(\xi,\eta)e^{\left\{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}}d\xi d\eta$$

or the Fourier transform of the equation

Fourier transform of the $U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}$ which is complex field just to the right of aperture multiplied by a quadratic phase factor

$$U\left(x,y\right)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\int\int_{-\infty}^{\infty}\left\{U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta$$

Seond form of the Fresnel diffraction integral

Where $r_{01}>>\lambda,\ \frac{x-\xi}{z}<1,\ \frac{y-\eta}{z}<1,$ or observation is in the near field of the aperture or Fresnel diffraction region and scalar theory approximation are assumed

Or we can use the transfer function approach:

$$H_F(f_X,f_Y)=e^{jkz}e^{-j\pi\lambda z\left(f_X{}^2+f_Y{}^2\right)}$$

4.5 2 Fresnel diffraction by a thin sinusoidal amplitude grating-Talbot images II

By omitting the constat term e^{jkz}, H_F is:

$$H_F(f_X, f_Y) = e^{-j\pi\lambda z\left(f_X{}^2 + f_Y{}^2\right)}$$

In any problem that deals with a purely periodic structure, the transfer function approach yeilds the simplest calcultions.

1) find the spatial frequency spectrum of the field transmitted by the aperture:

$$t_A(\xi,\eta) = \frac{1}{2}\left[1 + m\cos(2\pi\xi / L\right]$$

$$\mathcal{F}\left\{t_A(\xi,\eta)\right\} = \frac{1}{2}\delta\left(f_X, f_Y\right) + \frac{m}{4}\delta\left(f_X + f_0, f_Y\right) + \frac{m}{4}\delta\left(f_X - f_0, f_Y\right); \text{ with } f_0 = \frac{1}{L}$$

$$\mathcal{F}\left\{t_A(\xi,\eta)\right\} = \frac{1}{2}\delta\left(f_X, f_Y\right) + \frac{m}{4}\delta\left(f_X + \frac{1}{L}, f_Y\right) + \frac{m}{4}\delta\left(f_X - \frac{1}{L_0}, f_Y\right)$$

The transfer function evaluated at $\left(f_X, f_Y\right) = \left(\pm\frac{1}{L}, 0\right)$ becomes

$H\left(\pm\frac{1}{L}, 0\right) = e^{-j\frac{\pi\lambda z}{L^2}}$ and it is unity at the origin. So after propagation of a distance z the Fourier transform of the field becomes:

$$\mathcal{F}\left\{U(x,y)\right\} = \frac{1}{2}\delta\left(f_X, f_Y\right) + \frac{m}{4}e^{-j\frac{\pi\lambda z}{L^2}}\delta\left(f_X + \frac{1}{L}, f_Y\right) + \frac{m}{4}e^{-j\frac{\pi\lambda z}{L^2}}\delta\left(f_X - \frac{1}{L_0}, f_Y\right)$$

$$U(x,y) = \mathcal{F}^{-1}\mathcal{F}\left\{U(x,y)\right\} = \frac{1}{2} + \frac{m}{4}e^{-j\frac{\pi\lambda z}{L^2}}e^{j\frac{2\pi x}{L}} + \frac{m}{4}e^{-j\frac{\pi\lambda z}{L^2}}e^{-j\frac{2\pi x}{L}}$$

4.5 2 Fresnel diffraction by a thin sinusoidal amplitude grating-Talbot images III

$$U(x,y)=\frac{1}{2}\left[1+me^{-j\frac{\pi\lambda z}{L^2}}\cos\left(\frac{2\pi x}{L}\right)\right]$$

$$I(x,y)=\frac{1}{4}\left[1+2m\cos\left(\frac{\pi\lambda z}{L^2}\right)\cos\left(\frac{2\pi x}{L}\right)+m^2\cos^2\left(\frac{2\pi x}{L}\right)\right]$$

Now consider 3 special cases for the observation distance:

1) $\frac{\pi\lambda z}{L^2}=2n\pi\rightarrow z=\frac{2nL^2}{\lambda}$ where $n=0,\pm1,\pm2$

$$I(x,y)=\frac{1}{4}\left[1+2m\cos\left(\frac{2\pi x}{L}\right)+m^2\cos^2\left(\frac{2\pi x}{L}\right)\right]=\frac{1}{4}\left[1+m\cos\left(\frac{2\pi x}{L}\right)\right]^2$$

this is perfect image of the grating. These images that are formed without aid of a lens are called "Talbot images" or "self-images".

2) $\frac{\pi\lambda z}{L^2}=(2n+1)\pi\rightarrow z=\frac{(2n+1)L^2}{\lambda}$ where $n=0,\pm1,\pm2$

$$I(x,y)=\frac{1}{4}\left[1-2m\cos\left(\frac{2\pi x}{L}\right)+m^2\cos^2\left(\frac{2\pi x}{L}\right)\right]=\frac{1}{4}\left[1-m\cos\left(\frac{2\pi x}{L}\right)\right]^2$$

This is also image of the grating with a 180^0 spatial phase shift or "contrast reversal". These too are called "Talbot images".

4.5 2 Fresnel diffraction by a thin sinusoidal amplitude grating-Talbot images IIII

3) $\frac{\pi\lambda z}{L^2} = (2n-1)\frac{\pi}{2} \rightarrow z = \frac{(n-\frac{1}{2})L^2}{\lambda}$ where $n = 0, \pm 1, \pm 2$ then

$\cos\left(\frac{\pi\lambda z}{L^2}\right) = 0$ Using $\cos^2\left(\frac{2\pi x}{L}\right) = \frac{1+\cos\left(4\pi x\right)}{2}$

$$I\left(x,y\right) = \frac{1}{4}\left[1+m^2\cos^2\left(\frac{2\pi x}{L}\right)\right] = \frac{1}{4}\left[\left(1+\frac{m^2}{2}\right)+\frac{m^2}{2}\cos\left(\frac{4\pi x}{L}\right)\right]$$

This is an image with twice frequency of the original grating and has reduced contrast (nisted of 1 and m we have 1 and $\frac{m^2}{2}$ and the background is now brighter by $+m^2/2$. This is called the "Talbot subimage".

For example for $m = 0.3$ we have $m^2/2 = 0.045$

Locations of Talbot image planes behind the grating