Linear systems

Linear system definition
Impulse response
Transfer function
Sampling theory

PHYS 258 Spring 2010 SJSU Eradat

Primary Goal

- Understand the way optical systems process the light
- Know all about the amplitudes and phases of the light waves reaching the image plane.
- A point source of light will be represented by a delta function
- An object will be represented by many point sources or delta functions
- Response of the optical system to these delta functions is subject of interest

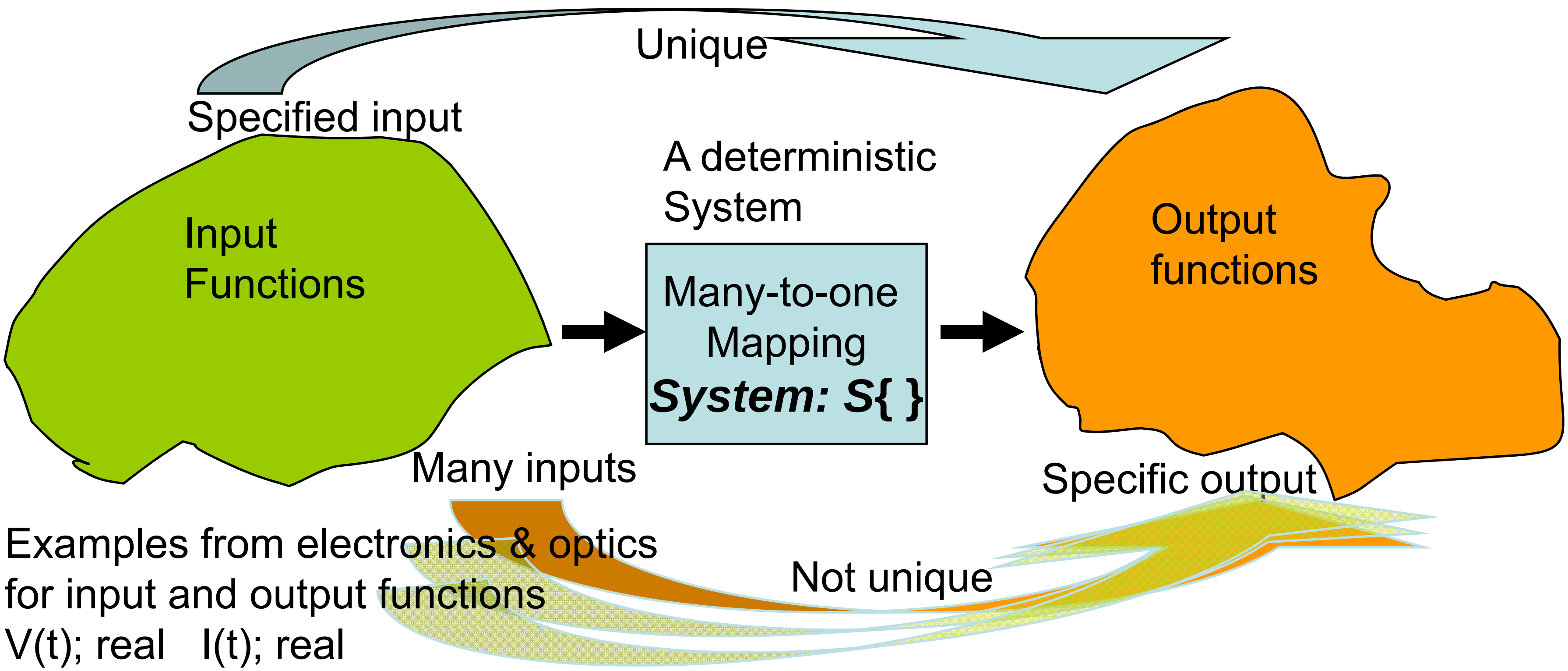
Linear systems

Output function=$S\{$Input function$\}$

$g_2(x_2, y_2) = S\{g_1(x_1, y_1)\}$

We need to specify action of S on g_1 to learn about g_2

We restrict ourselves to linear systems



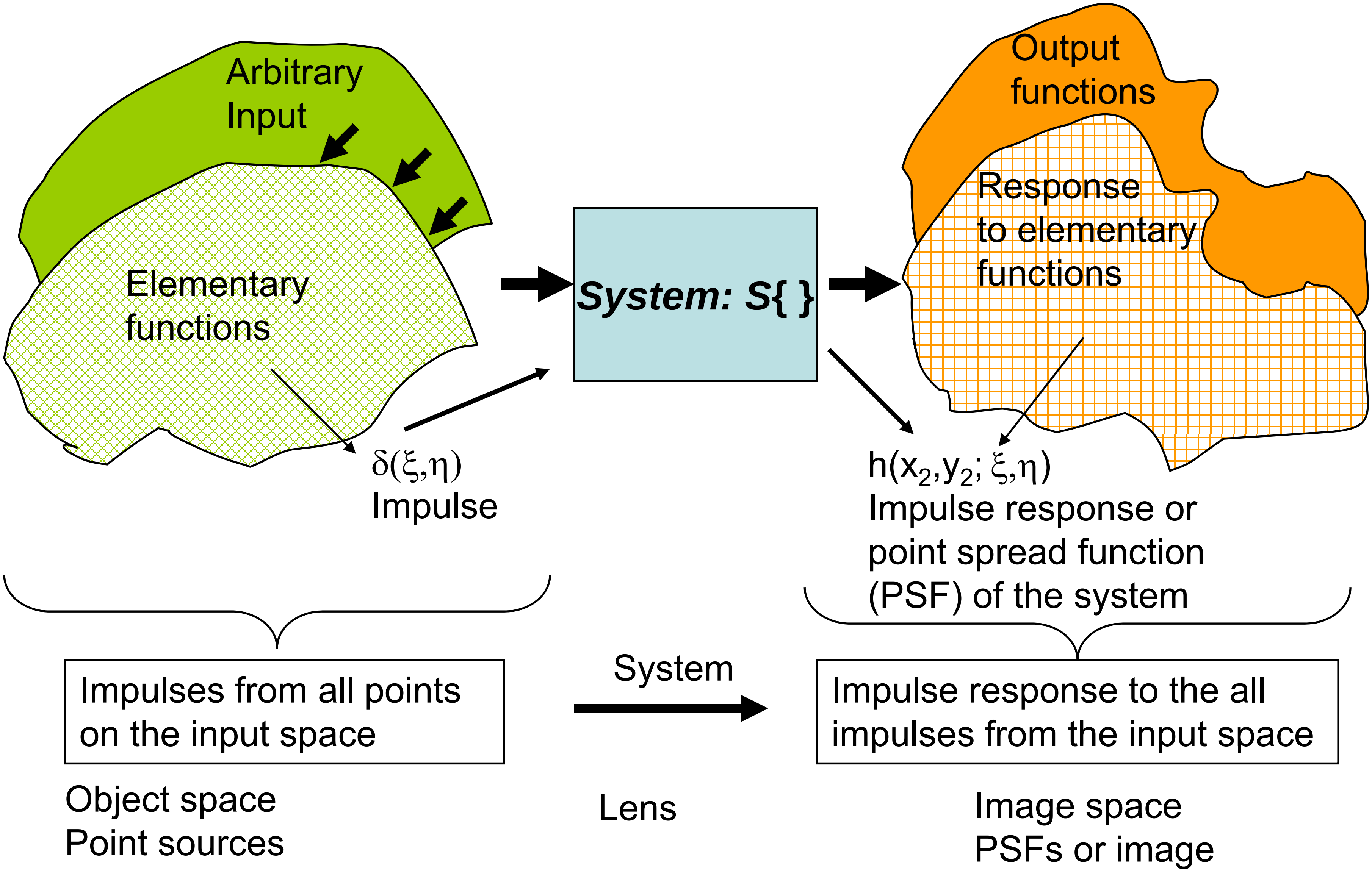
Examples from electronics & optics for input and output functions

V(t); real   I(t); real

Intensity(x,y); real

Field amplitude(x,y); complex

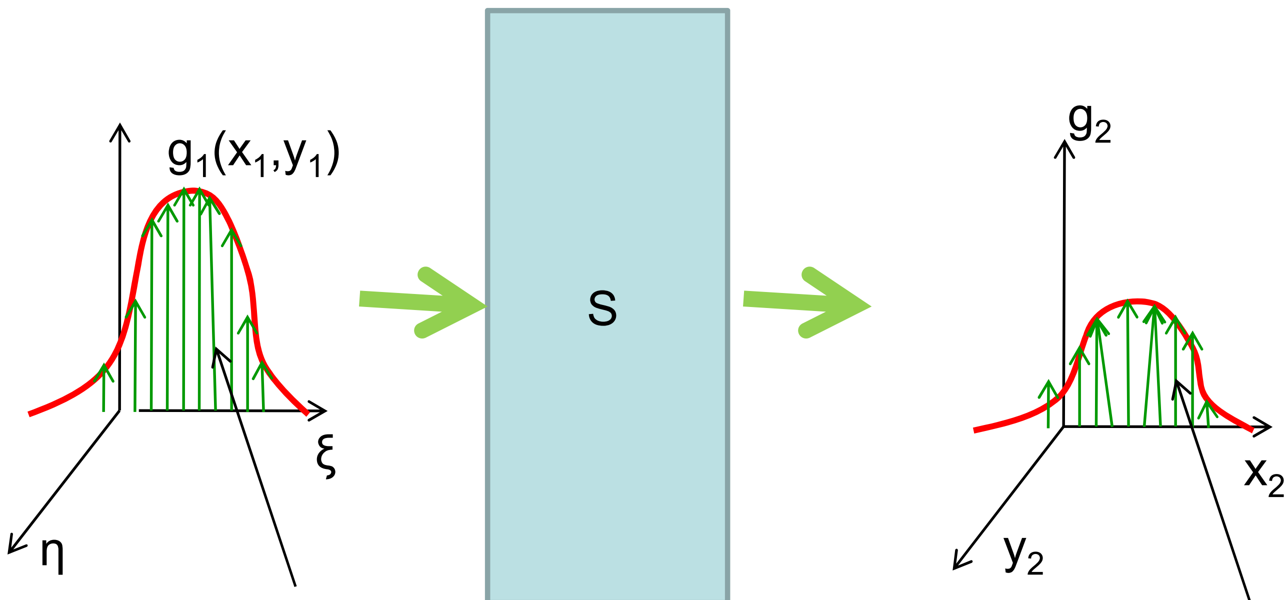
Impulse response or Point spread function for linear systems



Linearity and Superposition Integral I

Goal : show that A linear system can be completely characterized by its response to impulses.

We decompose an input signal to many elementary (δ) functions:



$$g_1(\xi,\eta)\delta(x_1-\xi;y_1-\eta)$$

$$g_1(\xi,\eta)S\{\delta(x_1-\xi;y_1-\eta)\}$$

Linearity and Superposition Integral II

We decompose an input signal to many elementary (δ) functions:

$$\underbrace{g_1(x_1, y_1)}_{\text{Input signal}} = \iint_{-\infty}^{\infty} \underbrace{g_1(\xi,\eta)}_{\substack{\text{Weighting}\\\text{factors}}} \; \underbrace{\delta(x_1-\xi; y_1-\eta)}_{\substack{\text{Elementary functions}\\\text{displaced } \delta \text{ functions}}} \, d\xi d\eta \leftarrow \begin{cases}\text{shifting property}\\\text{of delta functions}\end{cases}$$

$$\underbrace{g_2(x_2, y_2)}_{\text{Output signal}} = S\left\{\iint_{-\infty}^{\infty} \underbrace{g_1(\xi,\eta)}_{\text{Just a number}} \, \delta(x_1-\xi; y_1-\eta) d\xi d\eta\right\}$$ using linearity

$S\{ap(x_1,y_1)+bq(x_1,y_1)\} = aS\{p(x_1,y_1)\}+bS\{q(x_1,y_1)\}$ we get

$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} \underbrace{g_1(\xi,\eta)}_{\text{Just a number}} \; \underbrace{S\{\delta(x_1-\xi; y_1-\eta)\}}_{\text{Impulse response}} d\xi d\eta$$

Impulse response or Point-Spread Function (PSF) is the response of the system at point (x_2, y_2) of the output space to a δ function input at point (ξ,η) of the input space: $\boxed{h(x_2, y_2; \xi, \eta) = S\{\delta(x_1-\xi; y_1-\eta)\}}$

Superposition Integral: $\boxed{g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi,\eta) h(x_2, y_2; \xi, \eta) d\xi d\eta}$

Note: the impulses should cover all the input plane (object space).

A linear system can be completely characterized by its response to impulses.

Invariant Linear systems
A subclass of linear systems I

1) Time-invariant linear electrical network:

the system's response $h(t;\tau)$ to a unit impulse generated at time τ and measured at tiome t depends only on $(t - \tau)$

2) Space-invariant linear imaging system (or isoplanatic system):

the system's impulse response $h(x_2, y_2; \xi, \eta)$, depends only on the distances $(x_2 - \xi)$, $(y_2 - \eta)$ so for such a system:

$$h(x_2, y_2; \xi, \eta) = h(x_2 - \xi,\ y_2 - \eta)$$

Physical meaning of isoplanatic: image of the point object changes location on the image plane not functional form (shape) OR we look like ourselves in the mirror.

Most systems are not isoplanatic but can be treated as piecewise space-invariant.

Invariant Linear systems a subclass of linear systems II

Superposition integral:

$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi, \eta) h(x_2, y_2; \xi, \eta) d\xi d\eta$$

for isoplanatic systems

$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} \underbrace{g_1(\xi, \eta)}_{\text{Input (object function)}} \underbrace{h(x_2 - \xi, y_2 - \eta)}_{\text{Impulse response of the system}} d\xi d\eta$$

This is the two-dimensional convolution of the input (object) function with the impulse response of the system or $g_2 = g_1 \otimes h$ or $g_2 = g_1 * h$

For the invariant linear systems the output function is the convolution of the input (object) function with the impulse response of the sytem (or point-spread function PSF of the system)

$g_2 = g_1 \otimes h$ or $g_2 = g_1 * h$

Transfer functions I

Convolution takes a simple form after the Fourier transformation

$$g_2(x_2,y_2)=\iint_{-\infty}^{\infty}\underbrace{g_1(\xi,\eta)}_{\text{Object function}}\ \underbrace{h(x_2-\xi,y_2-\eta)}_{\text{Impulse response of the system}}d\xi d\eta=g_1*h$$

$$\mathcal{F}\left\{g_2(x_2,y_2)\right\}=\mathcal{F}\left\{\iint_{-\infty}^{\infty}g_1(\xi,\eta)h(x_2-\xi,y_2-\eta)d\xi d\eta\right\}=\mathcal{F}\left\{g_1*h\right\}$$

$$\underbrace{G_2(f_X,f_Y)}_{\text{Output spectrum}}=\overbrace{H(f_X,f_Y)\underbrace{G_1(f_X,f_Y)}_{\text{Input spectrum}}}^{\text{Simple multiplication}}$$

Where H is the <u>Fourier transform of the impulse response</u>.

$$H(f_X,f_Y)=\iint_{-\infty}^{\infty}h(\xi,\eta)e^{-j2\pi(f_X\xi+f_Y\eta)}d\xi d\eta$$

also known as <u>transfer function</u> of the system.

The transfer function represent <u>effects of the system</u> (on a impulse) in the <u>freuency domain</u>. We succeeded to reduce a convolution to simpler operations: Fourier transform $\rightarrow$ multiplication $\rightarrow$ inverse Fourier

Transfer functions II

The relation: $H(f_X, f_Y) = \iint_{-\infty}^{\infty} h(\xi,\eta) e^{-j2\pi(f_X\xi + f_Y\eta)} d\xi d\eta$

suggests that for linear-<u>invariant</u> (only) systems we may use exponential functions instead of the δ functions to decompose the input to elementay components.

In that case:

$$\underbrace{H(f_X, f_Y)}_{\text{System}} \; \underbrace{G_1(f_X, f_Y)}_{\text{Eigenfunctions of the linear invariant system}} = \overbrace{\underbrace{\text{A complex number}}_{\text{Eigenvalue}}}^{\text{Amplitude and phase shift}} \; \underbrace{G_1(f_X, f_Y)}_{\text{Eigenfunction}}$$

Complex exponential functions of the Fourier representation of the g_1

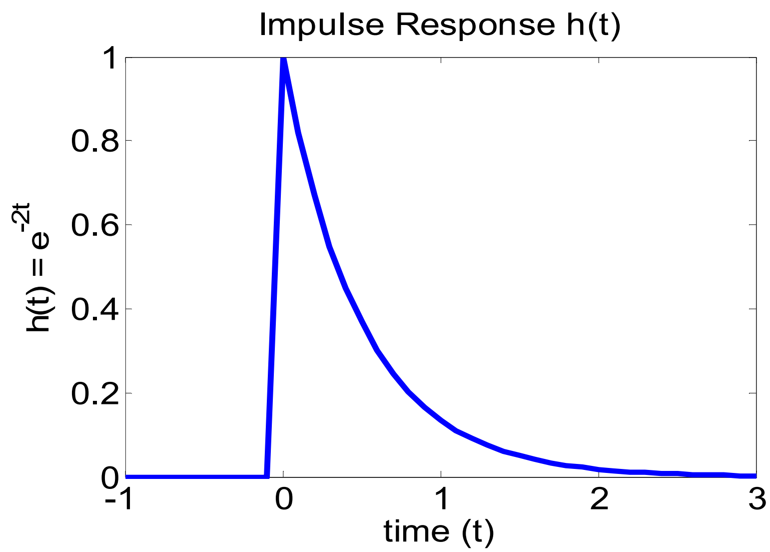
$$\underbrace{G_1(f_X, f_Y)}_{\text{Fourier transform of } g_1} = \mathcal{F}\{g_1(\xi,\eta)\} = \iint_{-\infty}^{\infty} \underbrace{g_1(\xi,\eta) e^{-j2\pi(f_X\xi + f_Y\eta)}}_{\text{Decomposition of } g_1 \text{ to complex exponential functions of various spatial frequency } (f_X, f_Y)} d\xi d\eta$$

$$\underbrace{H(f_X, f_Y)}_{\text{Transfer function}} = \underbrace{\iint_{-\infty}^{\infty} h(\xi,\eta) e^{-j2\pi(f_X\xi + f_Y\eta)} d\xi d\eta}_{\text{Fourier transform of the impulse response}}$$

See the references for the time-varying electrical and space-variant optical systems.

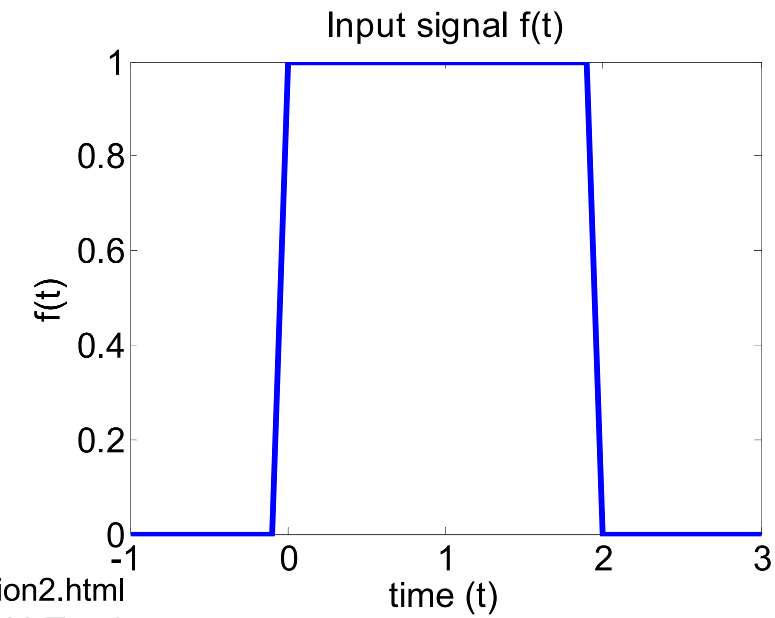
The convolution integral and its evaluation I

Impulse response of a system $h(t) = e^{-2t}$



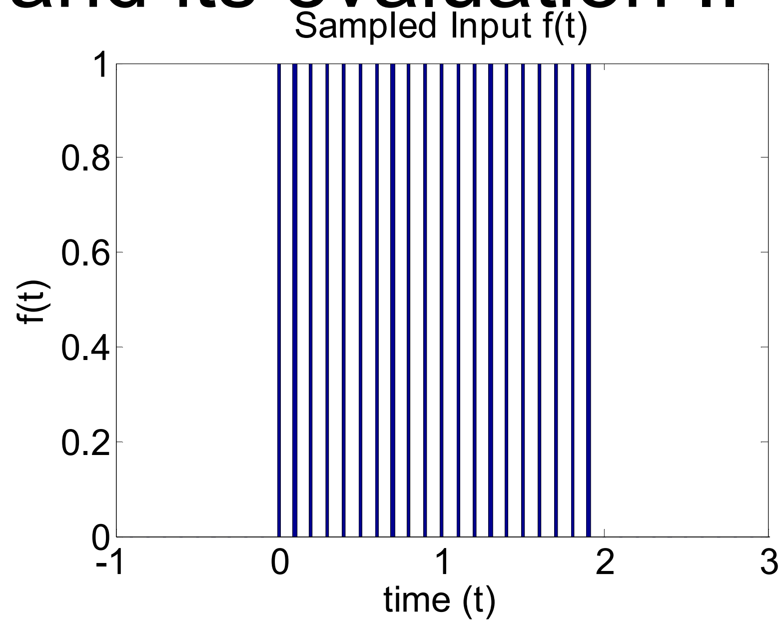
We want to find the response of the system

to the signal given by: $f(t) = \begin{cases} 1 & 0 < t < 2 \\ 0 & otherwise \end{cases}$



Ref. http://www.swarthmore.edu/NatSci/echeeve1/Ref/Convolution/Convolution2.html

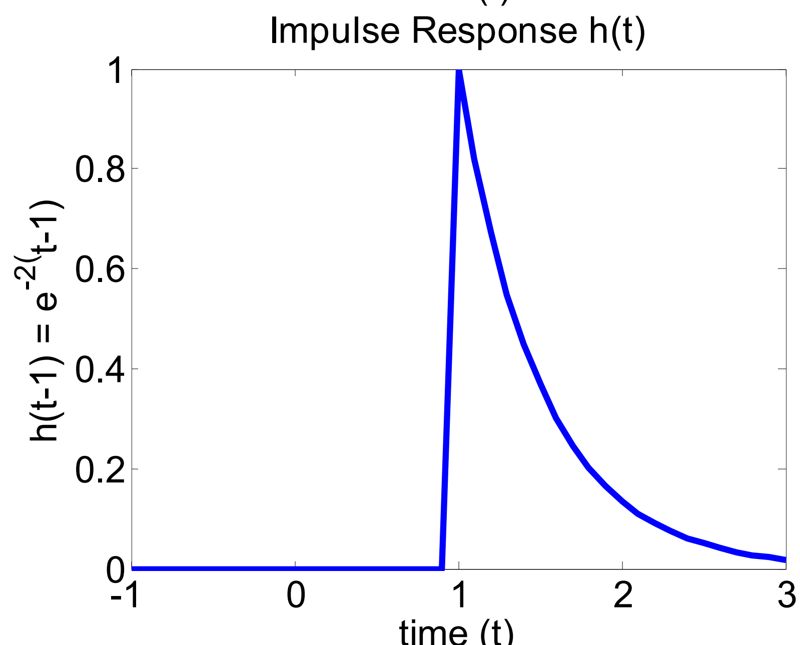
The convolution integral and its evaluation II



Next we find the impulse of the system to each of these samples

For the ith sample the impulse will be

$f(t_i)h(t-t_i)$



For example at $t_i = 1$ the impulse is

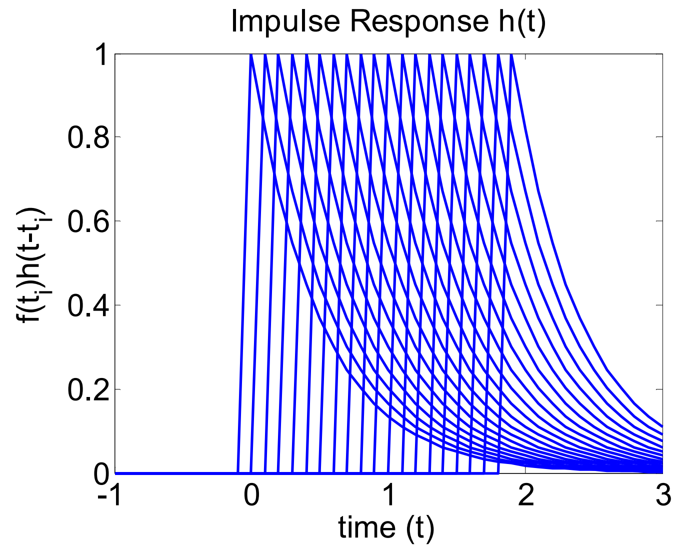
$f(1) \times h(t-1) = 1 \times e^{-2(t-1)}$

The convolution integral and its evaluation III

Next we find the impulse of the system to each of these samples

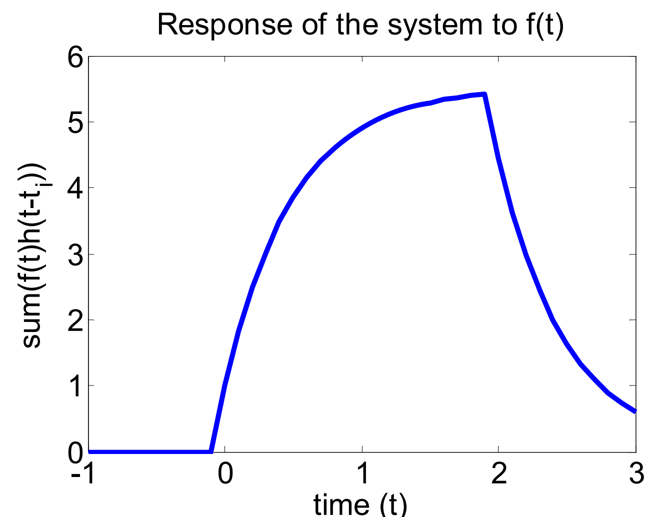
For the ith sample the impulse will be

$f(t_i)h(t-t_i)$



And the response at time t is the sum of individual responses to each sample before that time.

$$y(t) = \sum_{t_i=-1}^{3} f(t_i) \times h(t-t_i) = \sum_{t_i=-1}^{3} f(t_i) \times e^{-2(t-t_i)}$$



Note: For these graphs I took $t=3$ that means all of the signal had hit the system by then. Had I chosen the $t=1$ the the sum would have included only the impulse responses up to $t=1$.

For image from an object the limits of integral are $(-\infty, +\infty)$ on xy plane.

MATLAB code for convolution integral

```
% An illustration based on
%www.swarthmore.edu/NatSci/echeeve1/Ref/Convolut
ion/Convolution2.html
% for explanation of meaning of convolution integral.
clear
%% Impulse response
t = -1:.1:3;
j=1;
for i=1:1:41
    if t(1,i) < 0
        h_t(j,i)=0;
    else
        h_t(j,i) = exp(-2*t(1,i));
    end
end
plot(t,h_t(j,:),'LineWidth',3)
set(gca,'XTick',-1:1:3,'FontSize',18)
set(gca,'XTickLabel',{'-1','0','1','2','3'},'FontSize',18);
xlabel('time (t)','FontSize',18)
ylabel('h(t) = e^-^2^t','FontSize',18)
title('Impulse Response h(t)','FontSize',18)

%% Input signal
for i=1:1:41
    if t(1,i) >= 0 && t(1,i) < 2
        f_t(1,i)=1;
    else
        f_t(1,i) = 0;
    end
end
plot(t,f_t,'LineWidth',3)
set(gca,'XTick',-1:1:3,'FontSize',18)
set(gca,'XTickLabel',{'-1','0','1','2','3'},'FontSize',18);
xlabel('time (t)','FontSize',18)
ylabel('f(t)','FontSize',18)
title('Input signal f(t)','FontSize',18)
```

```
%% Sampling of the input signal
t = -1:.1:3;
for i=1:1:41
    if t(1,i) >= 0 && t(1,i) < 2
        f_t(1,i)=1;
    else
        f_t(1,i) = 0;
    end
end
bar(t,f_t,0.2)
set(gca,'XTick',-1:1:3,'FontSize',18)
set(gca,'XTickLabel',{'-1','0','1','2','3'},'FontSize',18);
xlabel('time (t)','FontSize',18)
ylabel('f(t)','FontSize',18)
title('Sampled Input f(t)','FontSize',18)

%% Impulse response of the sample at t=1
t = -1:.1:3;
ti = 0:.1:2;
j=11;
for i=1:1:41
    if (t(1,i)-ti(1,11)) < 0
        h_t(j,i)=0;
    else
        h_t(j,i) = exp(-2*(t(1,i)-ti(1,11)));
    end
end
plot(t,h_t(j,:),'LineWidth',3)
set(gca,'XTick',-1:1:3,'FontSize',18)
set(gca,'XTickLabel',{'-1','0','1','2','3'},'FontSize',18);
xlabel('time (t)','FontSize',18)
ylabel('h(t-1) = e^-^2^(t-1)','FontSize',18)
title('Impulse Response h(t)','FontSize',18)
```

```
%% Impulse responses of the samples
clear
t = -1:.1:3;
ti = 0:.1:1.9;
for j=1:1:20
    for i=1:1:41
        if (t(1,i)-ti(1,j)) < 0
            h_t(j,i)=0;
        else
            h_t(j,i) = exp(-2*(t(1,i)-ti(1,j)));
        end
    end
    plot(t,h_t(j,:),'LineWidth',2)
    hold on

end
set(gca,'XTick',-1:1:3,'FontSize',18)
set(gca,'XTickLabel',{'-1','0','1','2','3'},'FontSize',18);
xlabel('time (t)','FontSize',18)
ylabel('f(t_i)h(t-t_i)','FontSize',18)
title('Impulse Response h(t)','FontSize',18)
hold off

%% Summing the impulse responses of the
samples
y_t=zeros(1,41);
for j=1:1:20
    y_t(1,:)=y_t(1,:)+h_t(j,:);
end
plot(t,y_t,'LineWidth',3)
set(gca,'XTick',-1:1:3,'FontSize',18)
set(gca,'XTickLabel',{'-1','0','1','2','3'},'FontSize',18);
xlabel('time (t)','FontSize',18)
ylabel('sum(f(t)h(t-t_i))','FontSize',18)
title('Response of the system to f(t)','FontSize',18)
```

Some frequently used functions

Rectangle function: $$\mathrm{rect}(x)=\begin{cases}1 & |x|<1/2\\ 1/2 & |x|=1/2\\ 0 & otherwise\end{cases}$$

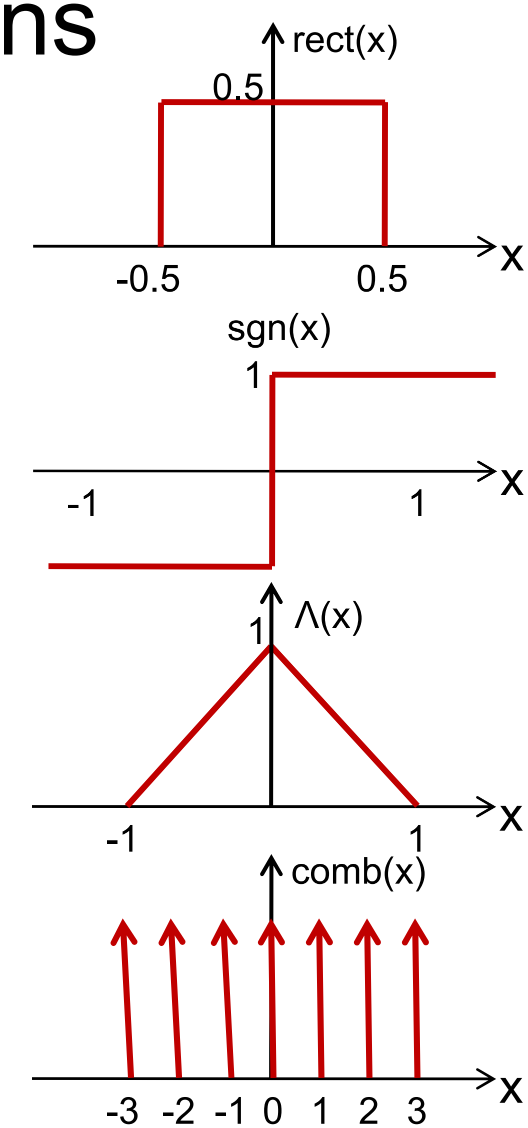
Sinc function: $$\mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$$

Signum function: $$\mathrm{sgn}(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$$

Triangle function: $$\Lambda(x)=\begin{cases}1-|x| & |x|\le 1\\ 0 & otherwise\end{cases}$$

Comb function: $$\mathrm{comb}(x)=\sum_{n=-\infty}^{\infty}\delta(x-n)$$

Circle function: $$\mathrm{circ}\left(\sqrt{x^2+y^2}\right)=\begin{cases}1 & \sqrt{x^2+y^2}<1\\ 1/2 & \sqrt{x^2+y^2}=1\\ 0 & otherwise\end{cases}$$



Useful Fourier transform pairs

$$\exp\left[-\pi\left(a^2x^2+b^2y^2\right)\right] \rightarrow \frac{1}{|ab|}\exp\left[-\pi\left(\frac{f_X^2}{a^2}+\frac{f_Y^2}{b^2}\right)\right]$$

$$\operatorname{rect}(ax)\operatorname{rect}(by) \rightarrow \frac{1}{|ab|}\operatorname{sinc}\left(\frac{f_X}{a}\right)\operatorname{sinc}\left(\frac{f_Y}{b}\right)$$

$$\Lambda(ax)\Lambda(by) \rightarrow \frac{1}{|ab|}\operatorname{sinc}^2\left(\frac{f_X}{a}\right)\operatorname{sinc}^2\left(\frac{f_Y}{b}\right)$$

$$\delta(ax,by) \rightarrow 1/|ab|$$

$$\exp\left[j\pi(ax+by)\right] \rightarrow \delta\left(f_X-a/2,\, f_y-b/2\right)$$

$$\operatorname{sgn}(ax)\operatorname{sgn}(by) \rightarrow \frac{ab}{|ab|}\frac{1}{j\pi f_X}\frac{1}{j\pi f_Y}$$

$$\operatorname{comb}(ax)\operatorname{comb}(by) \rightarrow \frac{1}{|ab|}\operatorname{comb}\left(\frac{f_X}{a}\right)\operatorname{comb}\left(\frac{f_Y}{b}\right)$$

$$\exp\left[j\pi\left(a^2x^2+b^2y^2\right)\right] \rightarrow \frac{1}{|ab|}\exp\left[-j\pi\left(\frac{f_X^2}{a^2}+\frac{f_Y^2}{b^2}\right)\right]$$

$$\exp\left[-\left(a|x|+b|y|\right)\right] \rightarrow \frac{1}{|ab|}\frac{2}{1+\left(2\pi f_X/a\right)^2}\frac{2}{1+\left(2\pi f_Y/b\right)^2}$$

Two-dimensional sampling theory I

$g(x, y)$ can be represented by its sampled values on a discrete (x, y) plane.

The closer the samples, the more accurate the representation.

Whittaker-Shannon sampling theorem:

For certain class of functions (known as ***bandlimitted*** functions) the reconstruction is accurate if the interval between samples is not greater than a certain limit.

Bandlimitted functions are the functions that their Fourier transform is nonzero only in a limitted frequency ragion R

A good reference online
http://graphics.cs.ucdavis.edu/~okreylos/PhDStudies/Winter2000/SamplingTheory.html

Two-dimensional sampling theory II

We sample a signal $g(x,y)$ with array of δ functions:

$$g_s(x,y) = \underbrace{comb\left(\frac{x}{X}\right)comb\left(\frac{y}{Y}\right)}_{2D \text{ array of } \delta \text{ functions}}\underbrace{g(x,y)}_{\text{Signal}} \quad \text{where}$$

$$comb\left(\frac{x}{X}\right) = X\sum_{n=-\infty}^{\infty}\delta\left(x-nX\right);\ comb\left(\frac{y}{Y}\right) = Y\sum_{m=-\infty}^{\infty}\delta\left(y-mY\right)\ \mathcal{F}\left\{comb\left(\frac{x}{X}\right)\right\} = |X|\,comb\left(Xf_X\right)$$

Taking Fourier transforms of the both sides we get:

$$\mathcal{F}\left\{g_s(x,y)\right\} = \mathcal{F}\left\{comb\left(\frac{x}{X}\right)comb\left(\frac{y}{Y}\right)g(x,y)\right\} \text{ with } \mathcal{F}\left\{g_s\right\} = G_s \text{ and } \mathcal{F}\left\{g\right\} = G$$

$$G_s\left(f_X, f_Y\right) = \mathcal{F}\left\{comb\left(\frac{x}{X}\right)comb\left(\frac{y}{Y}\right)\right\} \otimes G\left(f_X, f_Y\right)$$

We used frequency convolution. If $\mathcal{F}\left\{g_1(x)\right\} = G_1(f_X)$ and $\mathcal{F}\left\{g_2(x)\right\} = G_2(f_X)$

$$\underbrace{g_1(x)\otimes g_2(x) \Leftrightarrow G_1(f_X)\ G_2(f_X)}_{\text{Space convolution}} \quad \text{also} \quad \underbrace{g_1(x)g_2(x) \Leftrightarrow \frac{1}{2\pi}G_1(f_X)\otimes G_2(f_X)}_{\text{Spatial frequency convolution}}$$

$$\mathcal{F}\left\{comb\left(\frac{x}{X}\right)comb\left(\frac{y}{Y}\right)\right\} = XYcomb(Xf_X)comb(Yf_Y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta\left(f_X - \frac{n}{X}, f_Y - \frac{m}{Y}\right)$$

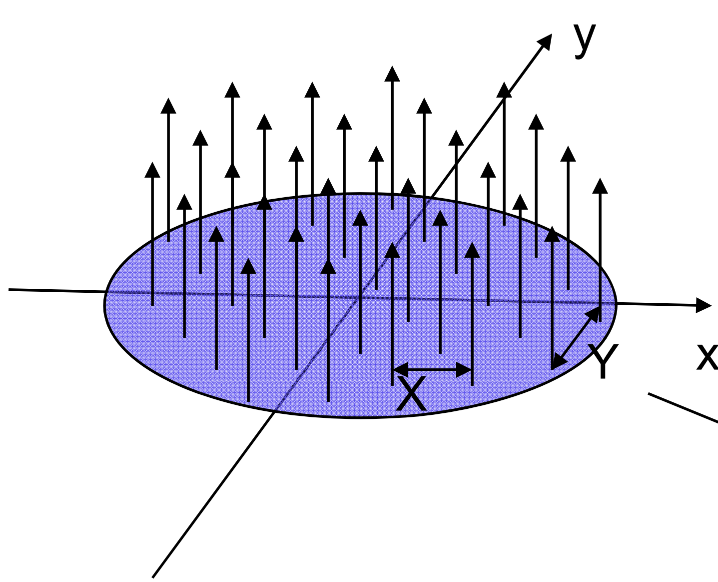
$$G_s\left(f_X, f_Y\right) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta\left(f_X - \frac{n}{X}, f_Y - \frac{m}{Y}\right)\otimes G\left(f_X, f_Y\right) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G\left(f_X - \frac{n}{X}, f_Y - \frac{m}{Y}\right)$$

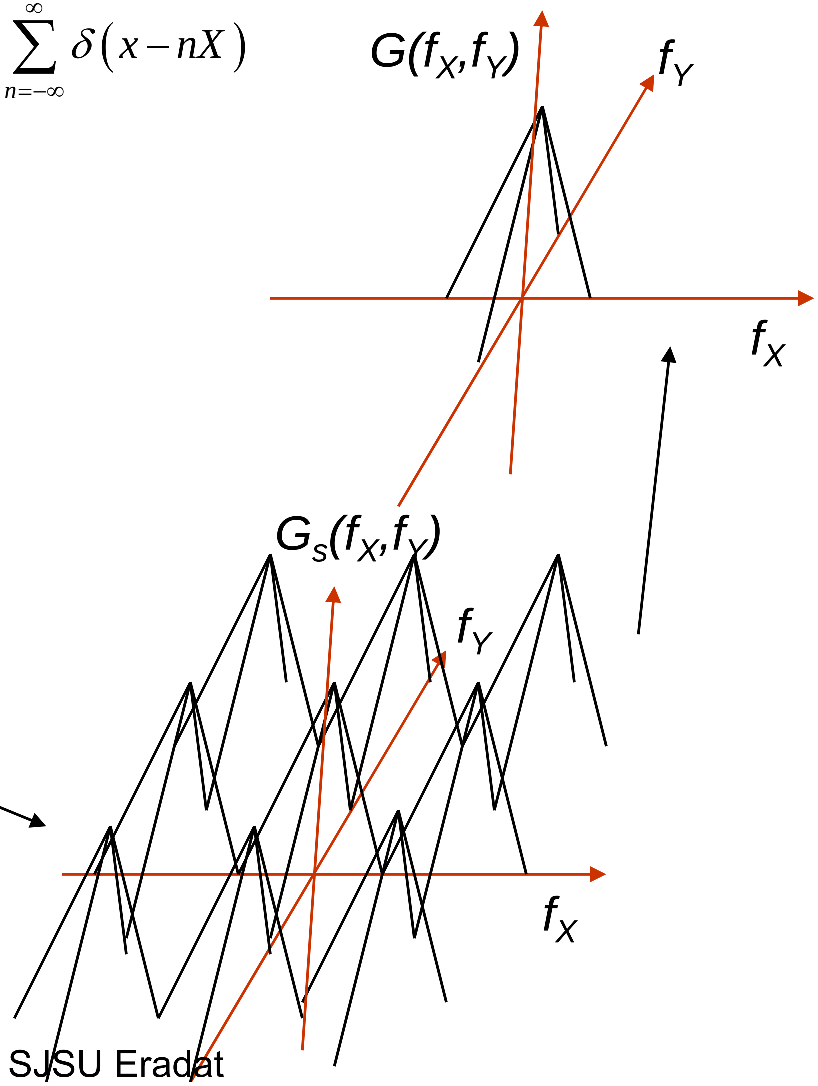
Two-dimensional sampling theory II

$$g_s(x,y) = comb\left(\frac{x}{X}\right) comb\left(\frac{y}{Y}\right) g(x,y)$$

where $comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$ and $comb\left(\frac{x}{X}\right) = X \sum_{n=-\infty}^{\infty} \delta(x-nX)$

$$G_s(f_X, f_Y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G\left(f_X - \frac{n}{X}, f_Y - \frac{m}{Y}\right)$$





Two-dimensional sampling theory III

We sample a signal $g(x,y)$ with array of δ functions:

$$g_s(x,y) = comb\left(\frac{x}{X}\right)comb\left(\frac{y}{Y}\right)g(x,y)$$

g is a ***bandlimitted*** function so its spectrum is nonzero over a certain region of (f_X, f_Y) plane which constructed about point $(\frac{n}{X}, \frac{m}{Y})$

When X and Y are small, the separation of the points are large so there is no overlap between the adjacent regions.

Now we can create g from g_s by sending the signal through a liniear invariant filter that allows only one signal around $n=0,\ m=0$ to pass and that is $G_s(f_X, f_Y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G\left(f_X - \frac{n}{X}, f_Y - \frac{m}{Y}\right)$

Maximum allowed separation between the samples to fully recover a signal

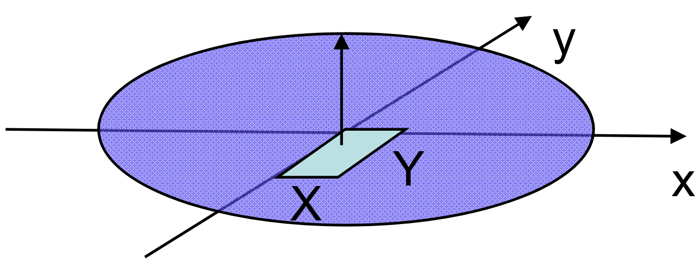
Width of G_s in f_X direction $= 2B_X$

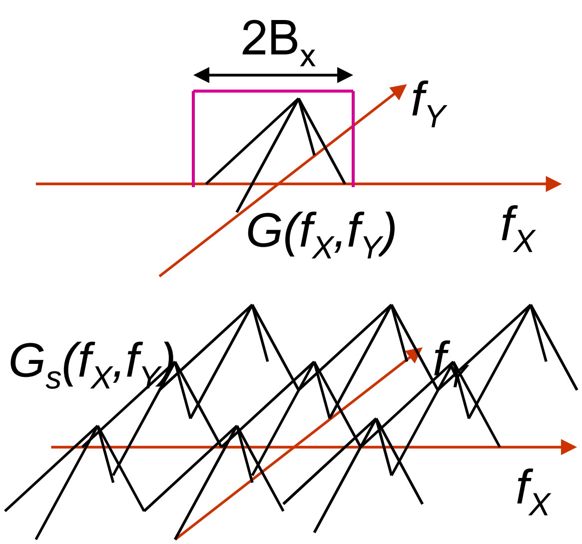
Width of G_s in f_Y direction $= 2B_Y$

$2B_X \times 2B_Y$ is the smallest rectangle that completely encloses a region R

Separation of various spectral elements in f_X and f_Y direction are $\frac{1}{X}$ and $\frac{1}{Y}$

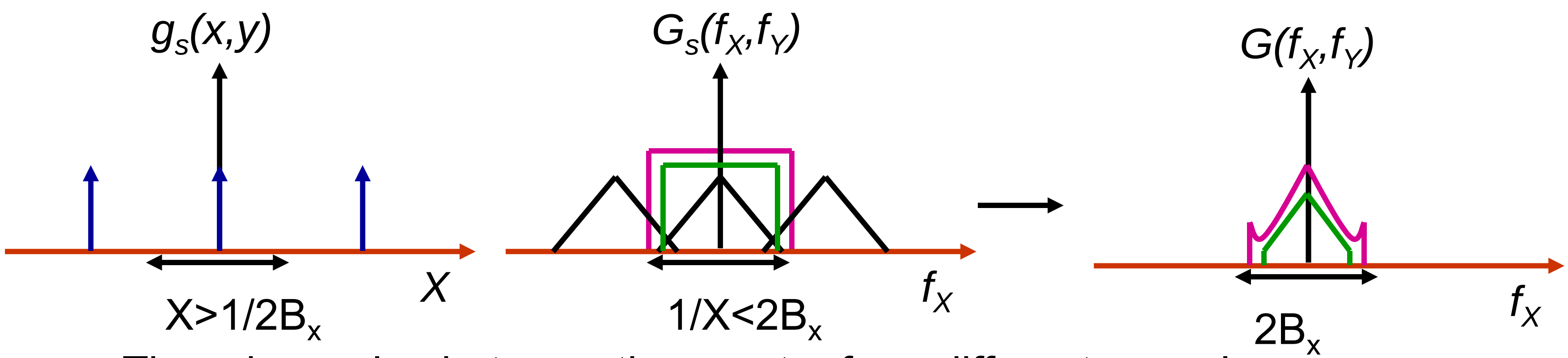
Then $\begin{cases} 2B_X \le \frac{1}{X} \rightarrow X \le \frac{1}{2B_X} \\ 2B_Y \le \frac{1}{Y} \rightarrow Y \le \frac{1}{2B_Y} \end{cases}$ assures the separation of the spectral regions.



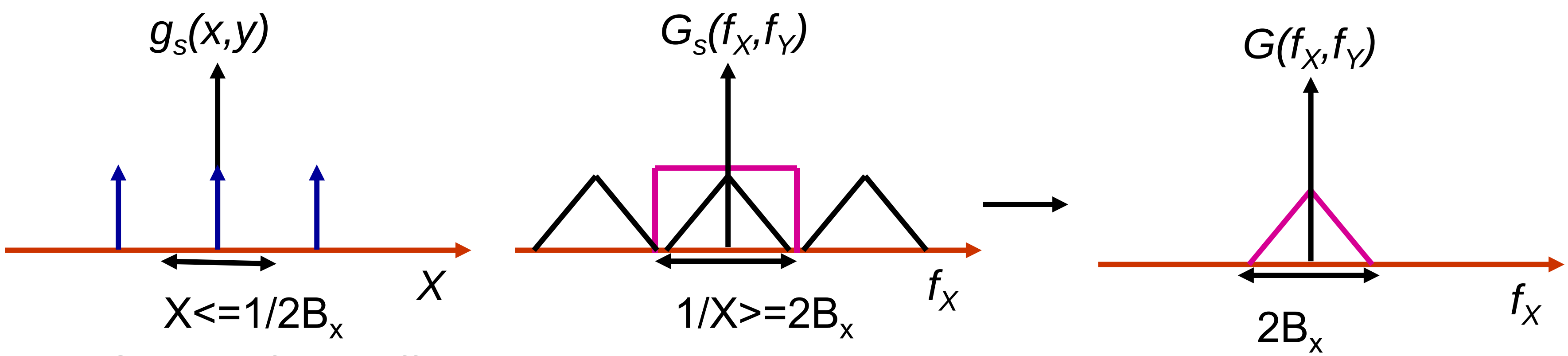


Proper choice of sampling interval and filter to recover $G(f_X,f_Y)$ from $G_s(f_X,f_Y)$

Let's only look at the x component



There is overlap between the spectra from different samples



Spectra from different samples are well separated

Choosing a proper transfer function for filtering $G(f_X, f_Y)$ from $G_s(f_X, f_Y)$ I

There is one transfer function that will always yeild a result if sampling requirements $\left(X \le \frac{1}{2B_X} \;\; \& \;\; Y \le \frac{1}{2B_Y} \right)$ are satisfied.The function is:

$$H(f_X, f_Y) = rect\left(\frac{f_X}{2B_X}\right) rect\left(\frac{f_Y}{2B_Y}\right)$$

we multiply this filter function by G_s.

$$G_s(f_X, f_Y) rect\left(\frac{f_X}{2B_X}\right) rect\left(\frac{f_Y}{2B_Y}\right) \equiv G\left(f_X, f_Y\right)$$

Multiplication in the frequency domain is a convolution in the spatial domain

$$G_s(f_X, f_Y) = \mathcal{F}\left\{g_s(x, y)\right\} = \mathcal{F}\left\{comb\left(\frac{x}{X}\right) comb\left(\frac{y}{Y}\right) g(x, y)\right\}$$

$$h(x, y) = \mathcal{F}^{-1}\left\{H(f_X, f_Y)\right\} = \mathcal{F}^{-1}\left\{rect\left(\frac{f_X}{2B_X}\right) rect\left(\frac{f_Y}{2B_Y}\right)\right\} = 4B_X B_Y \operatorname{sinc}\left(2B_X x\right) \operatorname{sinc}\left(2B_Y y\right)$$

h is impulse response of the filter

$$\left[comb\left(\frac{x}{X}\right) comb\left(\frac{y}{Y}\right) g(x, y)\right] \otimes h(x, y) = g(x, y)$$

Choosing a proper transfer function for filtering $G(f_X, f_Y)$ from $G_s(f_X, f_Y)$ II

Now we replace

$$\begin{cases} h(x,y) = \mathcal{F}^{-1}\left\{ rect\left(\frac{f_X}{2B_X}\right) rect\left(\frac{f_Y}{2B_Y}\right)\right\} = 4B_X B_Y \operatorname{sinc}\left(2B_X x\right)\operatorname{sinc}\left(2B_Y y\right) \quad \text{and} \\ comb\left(\frac{x}{X}\right) comb\left(\frac{y}{Y}\right) g(x,y) = XY \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g(nX, mY)\delta\left(x-nX, y-mY\right) \end{cases}$$

in $\left[comb\left(\frac{x}{X}\right) comb\left(\frac{y}{Y}\right) g(x,y)\right] \otimes h(x,y) = g(x,y)$ to get

$$g(x,y) = 4B_X B_Y XY \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g(nX, mY) \underbrace{\operatorname{sinc}[2B_X\left(x-nX\right)]}_{\text{sifted by } \delta \text{ function}} \underbrace{\operatorname{sinc}[2B_Y\left(y-mY\right)]}_{\text{sifted by } \delta \text{ function}}$$

When X and Y are the maximum allowable i.e $1/2B_X$ and $1/2B_Y$ we get

$$\boxed{g(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g\left(\frac{n}{2B_X}, \frac{m}{2B_Y}\right) \operatorname{sinc}\left[2B_X\left(x-\frac{n}{2B_X}\right)\right] \operatorname{sinc}\left[2B_Y\left(y-\frac{m}{2B_Y}\right)\right]}$$

This fundamental result is called Wittaker-Shannon sampling theorem.

Wittaker-Shannon Sampling theorem

- Exact recovery of a bandlimitted function can be achieved from an appropriately spaced rectangular array of its sampled values.

$$g(x,y)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\underbrace{g\left(\frac{n}{2B_X},\frac{m}{2B_Y}\right)}_{\text{Weights}}\underbrace{\operatorname{sin}c[2B_X\left(x-\frac{n}{2B_X}\right)]\operatorname{sin}c[2B_Y\left(y-\frac{m}{2B_Y}\right)]}_{\text{Elementary components}}$$

- Instead of comb functions we have sinc functions.
- This is not the only form of the sampling theorem.

Bessel functions I

The Bessel functions or cylinder functions or cylinderical harmonics

of the first kind, $J_n(x)$, are defined as the solutions to the

Bessel differential equation: $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$

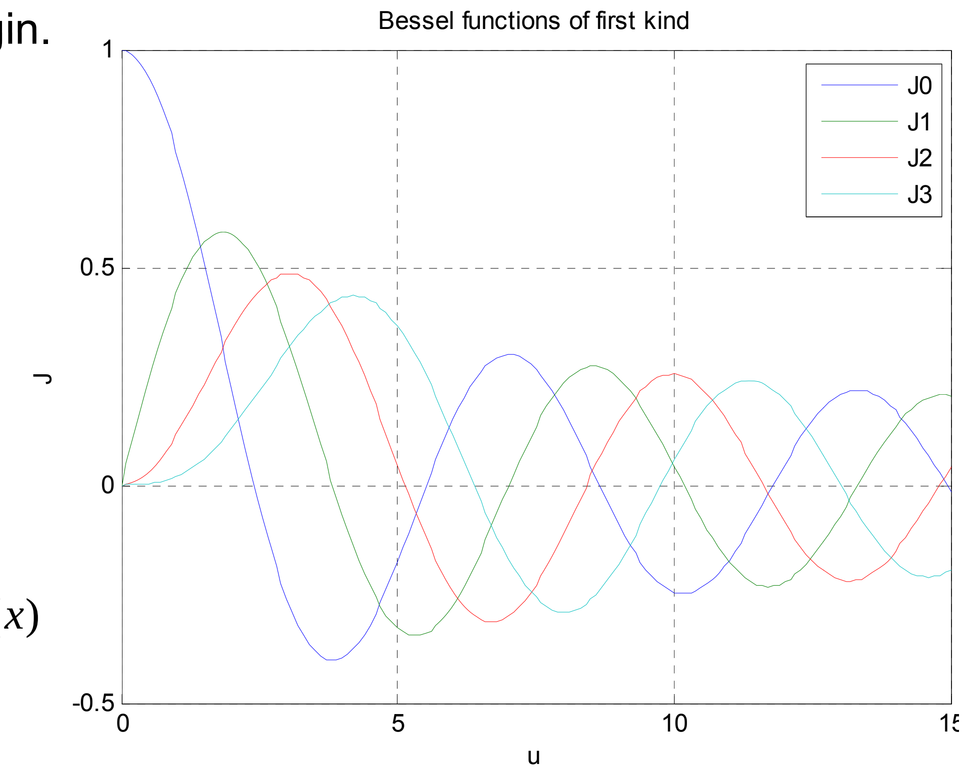
These functions are nonsingular at the origin.

$$J_m(x) \begin{cases} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+|m|} l!(|m|+l)!} x^{2l+|m|} & |m| \neq \frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \sin x & m = \frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \cos x & m = -\frac{1}{2} \end{cases}$$

$$J_{-m}(x) = (-1)^m J_m(x) \quad m = 0, 1, 2, 3, \ldots$$

A derivative identity: $\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x)$

An integral identity: $\int_0^u u' J_0(u') du' = u J_1(u)$



Bessel function addition theorem: $J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y) J_{n-m}(z)$

$$\sum_{k=-\infty}^{\infty} J_k(x) = 1; \qquad e^{iz\cos\theta} = J_0(z) + 2\sum_{-\infty}^{\infty} j^n J_n(z) \cos(n\theta)$$

There are more of these identities. Check you favorite math handbook.

Bessel functions II

Various integrals expressed in terms of the Bessel functions:

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(z\ sin\theta - n\theta)d\theta$$ Bessel's first integral

$$\mathrm{J}_n(z) = \frac{i^{-n}}{\pi}\int_0^\pi e^{iz\cos\theta}\cos(n\theta)d\theta$$

$$J_n(z) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{iz\cos\theta}e^{in\theta}d\theta \rightarrow j_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{ia\cos\theta}d\theta = \sum_{k=0}^{\infty}(-1)^k\frac{\left(Z^2/4\right)^k}{\left(k!\right)^2}$$

$$J_n(z) = \frac{2}{\pi}\frac{z^n}{\left(2n-1\right)!!}\int_0^{\pi/2}\sin^{2n}u\cos(z\cos u)du$$ for $n = 1, 2, \ldots$

$$J_n(x) = \frac{1}{2\pi i}\int_\gamma e^{\frac{x}{2}\frac{z-1}{z}}z^{-n-1}dz \quad for\ n > -\frac{1}{2}$$

The Bessel functions are normaized: $\int_0^\infty J_n(x)dx = 1$ for $n = 0, 1, 2, \ldots$

Integrals involving $J_1(x)$: $\int_0^\infty \left[\frac{J_1(x)}{x}\right]^2 dx = \frac{4}{3\pi}$ and $\int_0^\infty \left[\frac{J_1(x)}{x}\right]^2 xdx = \frac{1}{2}$

Transform of a circularly symmetric function I

Most apertures and lenses have circular symmetry for example

$g(x,y)=\{^{1 \quad \sqrt{x^2+y^2}\le a}_{0 \quad \sqrt{x^2+y^2}>a}$ expresses a circular aperture with radius of a.

The circular symmetry justifies usage of cylindical coordinates.

$$x = r\cos\theta; \quad y = r\sin\theta; \quad r=\sqrt{x^2+y^2}; \quad \theta=\tan^{-1}(y/x)$$

$$f_x = \rho\cos\phi; \quad f_y = \rho\sin\phi; \quad \rho = \sqrt{f_X^2+f_Y^2}; \quad \phi = \tan^{-1}(f_X/f_Y)$$

$$dxdy = rdrd\theta; \quad df_X df_Y = \rho d\rho d\phi;$$

$$\mathcal{F}\{g(x,y)\} = G(f_X, f_Y) = \int\int_{-\infty}^{\infty} g(x,y) e^{-j2\pi(f_X x + f_Y y)} dxdy$$

Now apply change of varaibles:

$$\mathcal{F}\{g(r,\theta)\} = G_0(\rho,\phi) = \int_0^{2\pi} d\theta \int_0^{\infty} g(r,\theta) e^{-j2\pi(\rho\cos\phi r\cos\theta + \rho\sin\phi r\sin\theta)} rdr$$

For circularly symmetric functions g is only function of r. So we write:

$$g(r,\theta) = g_R(r)$$

$$G_0(\rho,\phi) = \int_0^{2\pi} d\theta \int_0^{\infty} g_R(r) e^{-j2\pi r\rho\cos(\theta-\phi)} rdr = \int_0^{\infty} g_R(r) rdr \int_0^{2\pi} e^{-j2\pi r\rho\cos(\theta-\phi)} d\theta$$

Transform of a circularly symmetric function II

$$G_0(\rho,\phi) = \int_0^\infty g_R(r) r dr \int_0^{2\pi} e^{-j2\pi r\rho\cos(\theta-\phi)} d\theta$$

this relation is correct for any value of ϕ including $\phi = 0$,

Value of the integral $\frac{1}{2\pi}\int_0^{2\pi} e^{-ja\cos(\theta)} d\theta \;= J_0(a)$ is own known as the

zeroth order Bessel function of the first kind.

With substituting $a = 2\pi r\rho$ and $\phi = 0$ we get:

$$\mathcal{B}(\rho) = G_0(\rho) = 2\pi\int_0^\infty r g_R(r) J_0(2\pi r\rho) dr \leftarrow$$ Fourier-Bessel transform, $\mathcal{B}$, or Hankel transform of zero order

The inverse Fourier-Bessel transform is then:

$$\mathcal{B}^{-1} g(r,\theta) = g_R(r) = 2\pi\int_0^\infty \rho G_0(\rho) J_0(2\pi r\rho) d\rho$$

Conclusions:

1) Fourier transform of a circularly symmetric function is a circularly summetric function itself.

2) There is no difference between the direct and inverse transform operations.

Transform of a circularly symmetric function III

Following the Fourier integral theorem. and simmilarity theorem, we get:

$\mathcal{B}\mathcal{B}^{-1}\left\{g_R(r)\right\} = \mathcal{B}^{-1}\mathcal{B}\left\{g_R(r)\right\} = \mathcal{B}\mathcal{B}\left\{g_R(r)\right\} = g_R(r) \;\leftarrow$ when $g_R(r)$ is continuous.

$$\mathcal{B}\left\{g_R(ar)\right\} = \frac{1}{a^2} G_0\left(\frac{\rho}{a}\right)$$

$\mathcal{B}$ for Fourier-Bessel transform.

All other Fourier transform theorems apply since this is just a special case of the general two-dimensional Fourier transforms.

Fourier transform of a circular aperture with radius a

$$g(x,y)=\begin{cases}1 & \sqrt{x^2+y^2}\le a\\ 0 & \sqrt{x^2+y^2}>a\end{cases}\rightarrow g_R(r)=\begin{cases}1 & r\le a\\ 0 & r>a\end{cases}$$ this is simmlar to $\mathrm{circ}(x,y)$

Substituting $g_R(r)$ in

$$G_0(\rho,\phi)=G_0(\rho)=2\pi\int_0^\infty r g_R(r)J_0(2\pi r\rho)dr$$

$$G_0(\rho)=2\pi\int_0^a rJ_0(2\pi r\rho)dr$$

Using the the integral identity: $\int_0^u u'J_0(u')du' = uJ_1(u)$

$$r'=2\pi r\rho \qquad r=0\rightarrow r'=0 \quad and\ r=a \quad r'=2\pi a\rho$$

$$G_0(\rho)=\frac{1}{2\pi\rho^2}\int_0^a 2\pi r\rho J_0(2\pi r\rho)d(2\pi r\rho)=\frac{1}{2\pi\rho^2}\int_0^{2\pi a\rho} r'J_0(r')dr'$$

$$G_0(\rho)=\frac{1}{2\pi\rho^2}2\pi a\rho J_1(2\pi a\rho)=a\frac{J_1(2\pi a\rho)}{\rho}=2\pi a^2\frac{J_1(2\pi a\rho)}{2\pi a\rho}$$ with $k_\alpha=2\pi\rho$

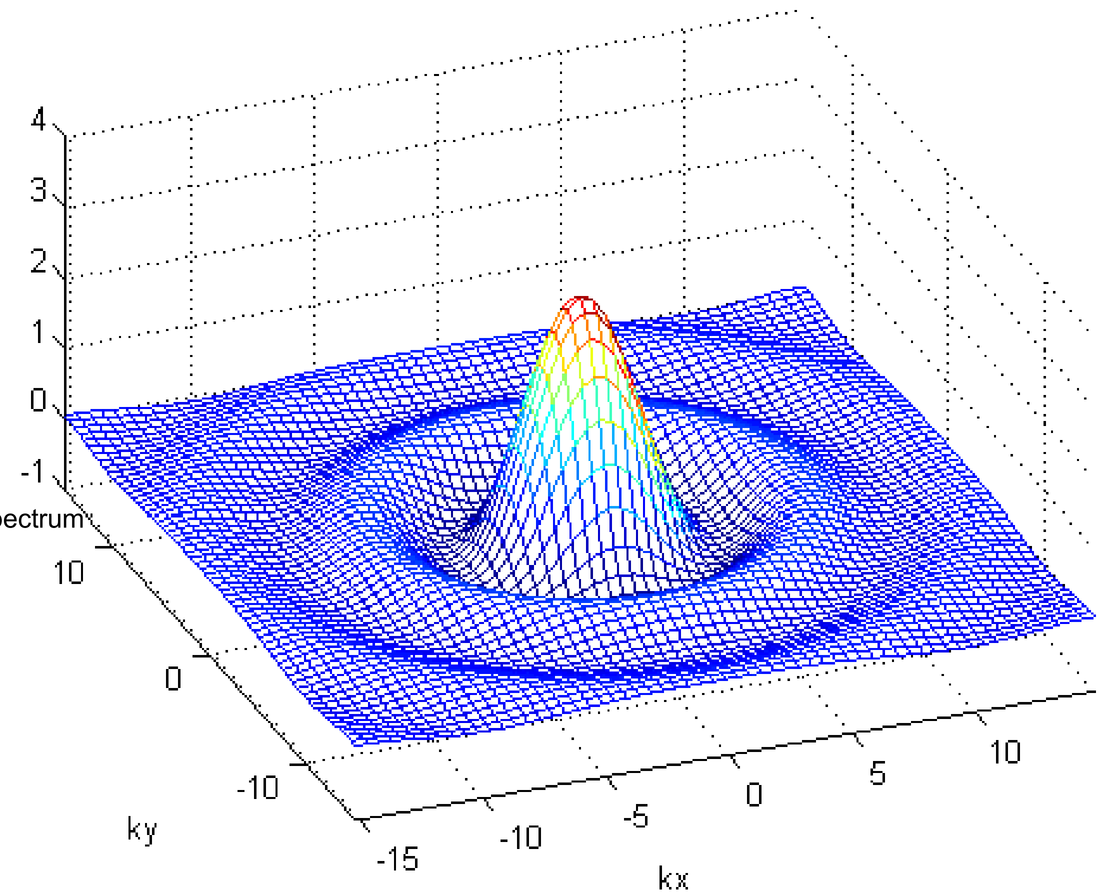
$$G_0(k_\alpha)=F\left(k_\alpha\right)=2\pi a^2\left[\frac{J_1(k_\alpha a)}{k_\alpha a}\right]$$ where J_1 is a first order Bessl function.

Circular aperture with Bessel functions in MATLAB

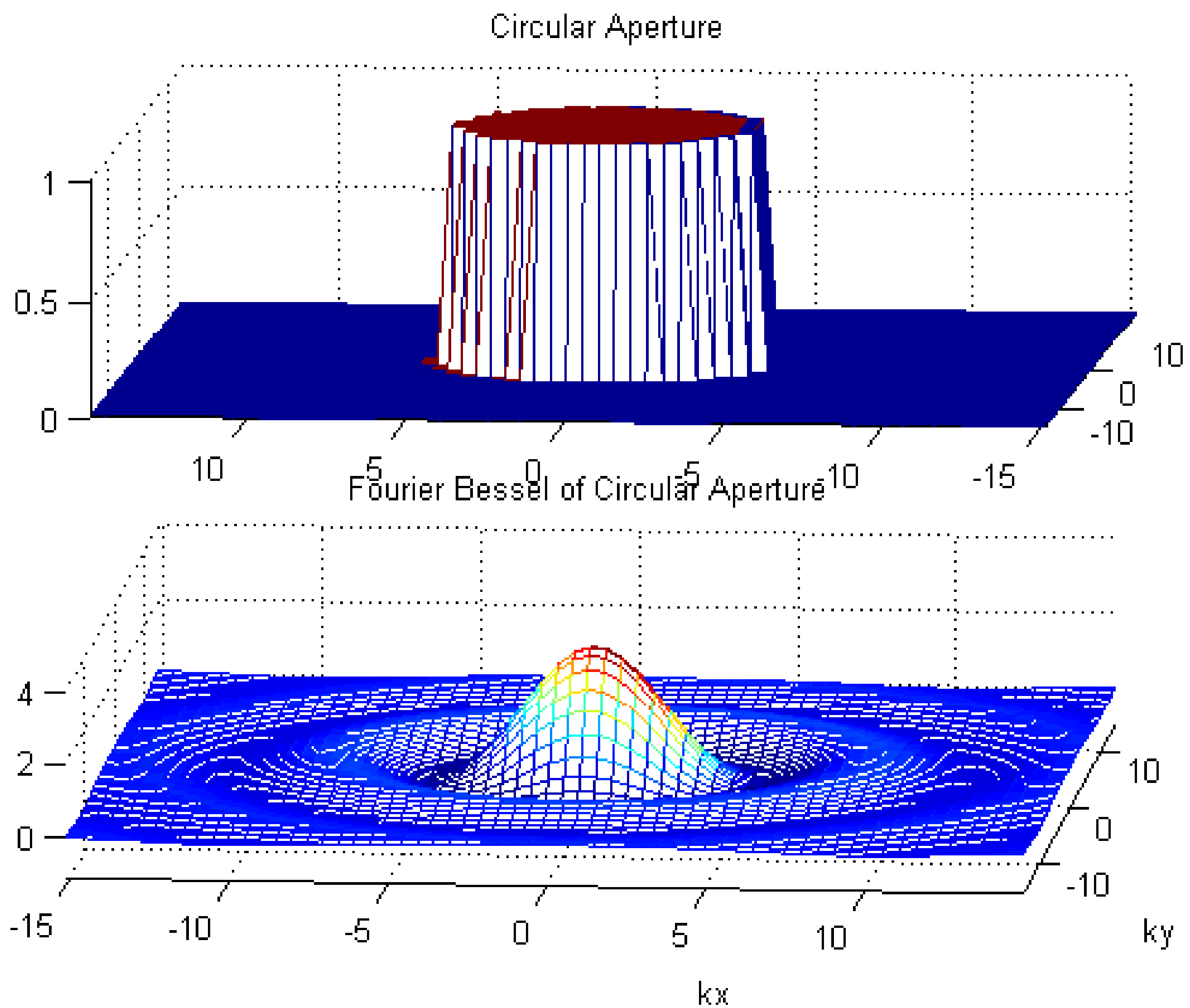
```
%PHYS 258 spring 07, Nayer Eradat
%A program to plot a circular aperture function
%and its Fourier transform using fft and shift fft function
x=(-2:0.05:2);
y=(-2:0.05:2);
A=y.'*x;
i_index=0;
for i=-2:0.05:2
   j_index=0;
   i_index=i_index+1;
   for j=-2:0.05:2
      j_index=j_index+1;
      r=sqrt(i^2+j^2);
      if r <=0.2
         A(i_index,j_index)=1;
      else A(i_index,j_index)=0;
      end
   end
end
subplot(2,1,1);
mesh(x,y,A); %3D plot
titel('Square aperture');
fft_v=abs(fft2(A));
fft_val=fftshift(fft_v);
%shift zero-frequency component to center of spectrum
subplot(2,1,2);
mesh(x,y,fft_val);
titel('fft of Square aperture');
```

$$g(x,y) = \begin{cases} 1 & \sqrt{x^2+y^2} \le a \\ 0 & \sqrt{x^2+y^2} > a \end{cases} \rightarrow g_R(r) = \begin{cases} 1 & r \le a \\ 0 & r > a \end{cases}$$

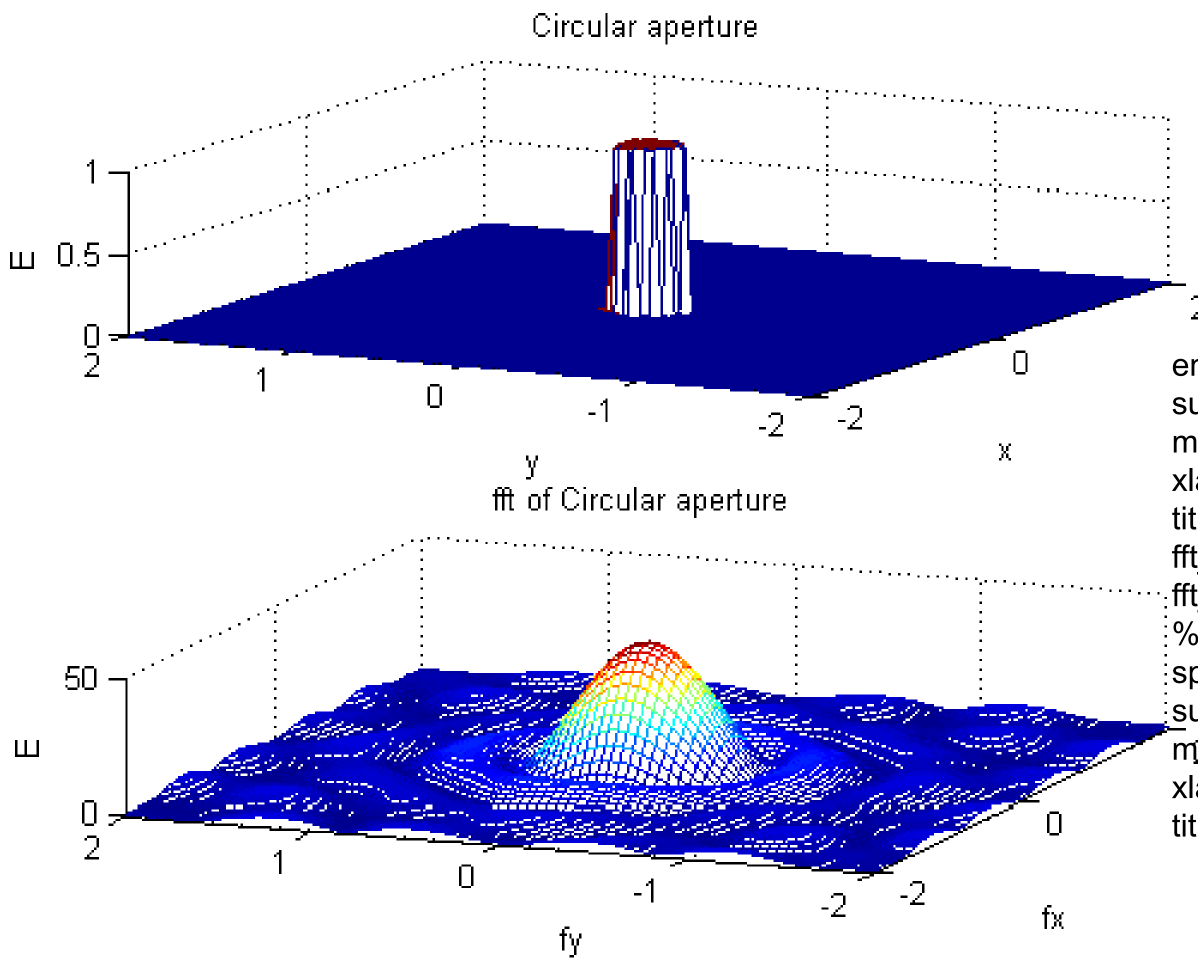
$$G_0(k_\alpha) = F(k_\alpha) = 2\pi a^2 \left[\frac{J_1(k_\alpha a)}{k_\alpha a} \right]$$



Circular aperture with Bessel functions in MATLAB



Circular aperture with FFT in MATLAB




```
%PHYS 258 spring 07, Nayer Eradat
%A program to plot a circular aperture function
%and its Fourier transform using fft and shift fft
function
x=(-2:0.05:2);
y=(-2:0.05:2);
A=y.'*x;
i_index=0;
for i=-2:0.05:2
    j_index=0;
    i_index=i_index+1;
    for j=-2:0.05:2
        j_index=j_index+1;
        r=sqrt(i^2+j^2);
        if r <=0.2
            A(i_index,j_index)=1;
        else A(i_index,j_index)=0;
        end
    end
end
subplot(2,1,1);
mesh(x,y,A); %3D plot
xlabel('x'); ylabel('y'); zlabel('E');
title('Circular aperture');
fft_v=abs(fft2(A));
fft_val=fftshift(fft_v);
%shift zero-frequency component to center of
spectrum
subplot(2,1,2);
mesh(x,y,fft_val);
xlabel('fx'); ylabel('fy'); zlabel('E');
title('fft of Circular aperture');
```

Transform of the Dirac Delta function

Homework 3

Problems 2.4, 2.8, 2.11, 2.12 from Goodman

1) Appodizing of apertures is done to reduce the effect of secondary diffraction peaks on the image plane. This is achieved by covering the aperture with an amplitude mask that drops off linearly from the center (Hecht P542). A typical amsk has equation $f_1(x) = \{^{L-x \quad 0<x<L}_{L+x \quad -L<x<0}$ and zero elsewhere.

A) Calculate the Fraunhopher diffraction field for a normally incident monochromatic light at the aperture. $E_0 e^{ikx}$ Note that the diffracted field is the Fourier transform of the aperture function.

B) Compare the diffracted field from $f_1(x)$ with $f_2(x) = \{^{E_0 \quad -L<x<L}_{0 \quad |x|>L}$ (a square aperture seen in previous problems) for the same input field. For what ratio of E_0 and L the amplitudes and Intensities at the center of both apertures are equal.

C) For the case mentioned in part B plot the Intensity and magnitude of diffracted field for both apertures between -4π and 4π and integrate the area under the intensity plot . Is there any power loss due to apodization?

2) Calculate convolution of a square function $f_2(x) = \{^{E_0 \quad -L<x<L}_{0 \quad |x|>L}$ with itself directly in space domain. Take the Fourier transform of the self-convoluted function. Then use the convolution theorem to confirm your result.

3) Two norrow slits located at $-d/2$ and $d/2$ from the center of coordinate system on a dark film. Prove that the superosition of the diffracted fields from the slits is $2\cos(kd/2)$. Assume the incident field at the slits is a plane wave with unit amplitude and no initial pahse.

4) The cylinder function is defined as $f(x,y)=\begin{Bmatrix} 1 & \sqrt{x^2+y^2} \le a \\ 0 & \sqrt{x^2+y^2} \ge a \end{Bmatrix}$. A) Show that Fourier transform of this function in cylinderical coordinates is $F\left(k_\alpha\right) = 2\pi a^2 \left[\dfrac{J_1(k_\alpha a)}{k_\alpha a}\right]$ where J_1 is a first order Bessl function. Use $x = r\cos\theta,\ y = r\sin\theta,$

$k_x = k_\alpha \cos\alpha,\ k_y = k_\alpha \sin\alpha \ dxdy = rdrd\theta$

B) Plot the $F(k_\alpha)$ using the derived formula (use 3D plot in MATHLAB).

C) Use FFT2 function and plot the Fourier transform of the cylinder function $f(x,y)=\begin{Bmatrix} 1 & \sqrt{x^2+y^2} \le a \\ 0 & \sqrt{x^2+y^2} \ge a \end{Bmatrix}$. Compare the results of B and C.

D) If you were to design a filter to eliminate the side lobes of the $F\left(k_\alpha\right)$ in the frequency domain what the filter's function in the frequency domain will be? What it will be in the space domain? (most of this problem has ben solved in lecture notes but it is a good exercise to work it out one more time since it is such an essential part of the optical systems with circular apertures.