Superposition of waves (review)

PHYS 168
Lasers
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Superposition of waves

- Superposition of waves is the common conceptual basis for some optical phenomena such as:
  - Polarization
  - Interference
  - Diffraction

- What happens when two or more waves overlap in some region of space.

- How the specific properties of each wave affects the ultimate form of the composite disturbance?

- Can we recover the ingredients of a complex disturbance?
Linearity and superposition principle

The scalar 3D wave equation \( \frac{\partial^2 \psi(r,t)}{\partial r^2} = \frac{1}{V^2} \frac{\partial^2 \psi(r,t)}{\partial t^2} \) is a linear differential equation (all derivatives appear in first power). So any linear combination of its solutions \( \psi(r,t) = \sum_{i=1}^{n} C_i \psi_i(r,t) \) is a solution.

Superposition principle: resultant disturbance at any point in a medium is the algebraic sum of the separate constituent waves.

We focus only on linear systems and scalar functions for now. At high intensity limits most systems are nonlinear.

Example: Intensity of a typical focused laser beam \( \approx 10^{10} \text{ W/cm}^2 \) compared to sunlight on earth \( \approx 10 \text{ W/cm}^2 \).

Electric field of the laser beam can trigger nonlinear phenomena.
Superposition of two waves

Two light rays with same frequency meet at point $p$ traveled by $x_1$ and $x_2$

$E_1 = E_{01} \sin(\omega t - (kx_1 + \varepsilon_1)) = E_{01} \sin(\omega t + \alpha_1)$

$E_2 = E_{02} \sin(\omega t - (kx_2 + \varepsilon_2)) = E_{02} \sin(\omega t + \alpha_2)$

Where $\alpha_1 = -(kx_1 + \varepsilon_1)$ and $\alpha_2 = -(kx_2 + \varepsilon_2)$

Magnitude of the composite wave is sum of the magnitudes at a point in space & time or: $E = E_1 + E_2 = E_0 \sin(\omega t + \alpha)$ where

$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos(\alpha_2 - \alpha_1)$ and $\tan \alpha = \frac{E_{01} \sin \alpha_1 + E_{02} \sin \alpha_2}{E_{01} \cos \alpha_1 + E_{02} \cos \alpha_2}$

The resulting wave has same frequency but different amplitude and phase.

$2E_{01}E_{02} \cos(\alpha_2 - \alpha_1)$ is the interference term

$\delta = \alpha_2 - \alpha_1$ is the phase difference.
Phase difference and interference

\[ \delta = \alpha_2 - \alpha_1 = (kx_1 + \varepsilon_1) - (kx_2 + \varepsilon_2) = \frac{2\pi}{\lambda} (x_1 - x_2) + (\varepsilon_1 - \varepsilon_2) = \delta_1 + \delta_2 \]

**Total** phase difference between the two waves has two different origins.

a) \[ \delta_2 = (\varepsilon_1 - \varepsilon_2) \] phase difference due to the **initial** phase of the waves.

Waves with constant initial phase difference are said to be **coherent**.

b) \[ \delta_1 = \frac{2\pi}{\lambda_0} n(x_1 - x_2) = k_0 \Lambda \] is phase difference due to the **Optical Path Difference** or **OPD**

\[ \Lambda = n(x_1 - x_2) \]

**Waves in-phase**: \[ \delta \equiv \alpha_2 - \alpha_1 = 0, \pm 2\pi, \pm 4\pi, \ldots \] then \( E_0 \) is maximum

**Waves out of phase**: \[ \delta \equiv \alpha_2 - \alpha_1 = \pm \pi, \pm 3\pi, \ldots \] then \( E_0 \) is minimum

**Waves in-phase** interfere **constructively** \( E = E_{\text{max}} = (E_{01} + E_{02})^2 \)

**Waves out of phase** interfere **destructively** \( E = E_{\text{min}} = (E_{01} - E_{02})^2 \)

If \( E_{01} = E_{02} \) then \( E_{\text{max}} = (2E_{01})^2 \) and \( E_{\text{min}} = 0 \)

\[ \frac{\Lambda}{\lambda_0} = \frac{x_1 - x_2}{\lambda} \rightarrow \text{number of waves in medium} = \text{number of waves in vacuum} \]
Addition of two waves with same frequency
Two waves with path difference

For two waves with no initial phase difference \((\varepsilon_1 = \varepsilon_2 = 0)\) but a path difference of \(\Delta x\) we have:

\[
E_1 = E_{01} \sin(\omega t - k(x + \Delta x)) = E_{01} \sin[\omega t + \alpha_1]
\]

\[
E_2 = E_{02} \sin(\omega t - kx) = E_{02} \sin[\omega t + \alpha_2]
\]

\[
\alpha_2 - \alpha_1 = k\Delta x
\]

The resulting wave is

\[
E = 2E_0 \cos\left(\frac{k\Delta x}{2}\right) \sin\left[\omega t - k\left(x + \frac{\Delta x}{2}\right)\right]
\]

Constructive interference: if \(\Delta x \ll \lambda\), or \(\Delta x \approx \pm 2m\lambda\) then the resulting amplitude is \(\sim 2E_0\)

Destructive interference: \(\Delta x \approx \pm (2m+1)\lambda\) then \(E \approx 0\)
Exercise

2.1) Plot $E_1$, $E_2$, $E_1 + E_2$, and $(E_1 + E_2)^2$ for the following two sinusoidal waves for $0 < x < 5\lambda$ with $\lambda = 500$ nm:

$E_1 = E_{01} \sin(\omega t - (kx + \varepsilon_1))$ and $E_2 = E_{02} \sin(\omega t - (kx + \varepsilon_2))$

a) same frequency, $E_{01} = E_{02} = 2$, zero initial phase, both forward.

b) same frequency, $E_{01} = E_{02} = 2$, $\varepsilon_1 = 0$, $\varepsilon_2 = \pi$, both forward.

c) same frequency, $E_{01} = E_{02} = 2$, $\varepsilon_1 = 0$, $\varepsilon_2 = \pi/2$, both forward.

d) same frequency, $E_{01} = E_{02} = 2$, $\varepsilon_1 = 0$, $\varepsilon_2 = \pi$, $E_1$ forward, $E_2$ backward.

e) same frequency, $E_{02} = 2E_{01} = 2$, $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, both forward.

f) same frequency, $E_{02} = 2E_{01} = 2$, $\varepsilon_1 = 0$, $\varepsilon_2 = \pi$, both forward.

g) Compare the results of direct superposition with the formula derived in text for case a in slide 5 (Notice the difference between $\arctan$ and $\arctan^2$ functions in MATLAB)
Phasors and complex number representation

- Each harmonic function is shown as a rotating vector (phasor)
  - projection of the phasor on the x axis is the **instantaneous value of the function**, 
  - length of the phasor is the maximum amplitude
  - angle of the phasor with the positive x direction is the **phase of the wave**.

\[
E = E_0 e^{i(\omega t + \alpha_1)}
\]
Example of superposition using phasors

\[ E_1(t) = E \cos(\omega t + \phi) \quad E_2(t) = E \cos(\omega t) \]

\[ \vec{E}_p = \vec{E}_1 + \vec{E}_2 \text{ a vector sum of } \vec{E}_1 \text{ and } \vec{E}_2 \]

Magntude of \( \vec{E}_p \) (from trianognometry)

\[ E_p^2 = E^2 + E^2 - 2E^2 \cos(\pi - \phi) \]

\[ E_p^2 = E^2 + E^2 + 2E^2 \cos \phi \]

Using \( 1 + \cos \phi = 2 \cos^2(\phi/2) \)

\[ E_p^2 = 2E^2(1 + \cos \phi) = 4E^2 \cos^2(\phi/2) \]

Amplitude of two identical waves interfering with phase difference of \( \phi \)

is independent of time:

\[ E_p = 2E \left| \cos \frac{\phi}{2} \right| \]
Superposition of many waves

Superposition of any number of coherent harmonic waves with a given frequency, \( \omega \) and traveling in the same direction leads to a harmonic wave of that same frequency.

\[
E = \sum_{i=1}^{N} E_{0i} \cos(\alpha_i \pm \omega t) = E_0 \cos(\alpha \pm \omega t)
\]

\[
E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2 \sum_{j>i}^{N} \sum_{i=1}^{N} E_{0i} E_{0j} \cos(\alpha_i - \alpha_j) \quad \text{and} \quad \tan \alpha = \frac{\sum_{i=1}^{N} E_{0i} \sin \alpha_i}{\sum_{i=1}^{N} E_{0i} \cos \alpha_i}
\]

\( \alpha_i = -(kx + \epsilon_i) \) and \( \alpha_j = -(kx + \epsilon_j) \)

For coherent sources \( \alpha_i = \alpha_j \) and \( E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2 \sum_{j>i}^{N} \sum_{i=1}^{N} E_{0i} E_{0j} = \left( \sum_{i=1}^{N} E_{0i} \right)^2 \)

For incoherent sources (random phases) the second term is zero.

Flux density for \( N \) equal-amplitude emmitters: \( \left( E_0^2 \right)_{\text{incoh}} = N E_{01}^2 \); \( \left( E_0^2 \right)_{\text{coh}} = \left( N E_{0i} \right)^2 \)
Exercise

2.2) Write a MATLAB routine to calculate the amplitude and phase of $N$ harmonic waves (cosine) with same frequencies but varying initial phase and amplitudes. Assume the wavelength is 500 nm and $V = c$

a) The program should read the phase and amplitude of the waves from a file that has two columns and $N$ rows. Test the program for the following waves $E_1 = 1$, $\varepsilon_1 = 0$, $E_2 = 1$, $\varepsilon_2 = \pi / 4$. Once made sure it is working, create a file with the following waves and plot their superposition from 0 to $5\lambda$.

$E_1 = 1$, $\varepsilon_1 = 0$, $E_2 = 1$, $\varepsilon_2 = 10$, $E_3 = 2$, $\varepsilon_3 = 20^0$, $E_4 = 3$, $\varepsilon_4 = 30^0$, $E_5 = 2$, $\varepsilon_5 = 40^0$, $E_6 = 1$, $\varepsilon_6 = 50^0$, $E_7 = 1$, $\varepsilon_7 = 60^0$

b) Next run the program for $N = 101$ and $\varepsilon_i = \varepsilon_1 + \frac{i}{100} \pi$, where $\varepsilon_1 = \frac{\pi}{2}$ and $E_i = 2$. This time create the phases and amplitudes inside the routine and don't read from a file.
Addition of waves: different frequencies I

Mathematics behind light modulation and light as a carrier of information. Two propagating waves are superimposed

\[ E_1 = E_{01} \cos(k_1 x - \omega_1 t) \]
\[ E_2 = E_{01} \cos(k_2 x - \omega_2 t) \]

\( k_1 > k_2 \) and \( \omega_1 > \omega_2 \) with equal amplitudes and zero initial phases

\[ E = E_1 + E_2 = E_{01} [\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)] \]

using \( \cos \alpha + \cos \beta = 2 \cos \frac{1}{2} (\alpha + \beta) \cos \frac{1}{2} (\alpha - \beta) \)

\[ E = 2E_{01} \cos \frac{1}{2} [(k_1 + k_2)x - (\omega_1 + \omega_2)t] \times \cos \frac{1}{2} [(k_1 - k_2)x - (\omega_1 - \omega_2)t] \]

Need to simplify this
Addition of waves: different frequencies II

\[ E = 2E_0 \cos \left[ k_m x - \omega_m t \right] \times \cos \left[ \bar{k}x - \bar{\omega}t \right] \]

with the following definitions:

Average angular frequency \( \equiv \bar{\omega} = \left( \omega_1 + \omega_2 \right) / 2 \)

Average propagation number \( \equiv \bar{k} = \left( k_1 + k_2 \right) / 2 \)

Modulation angular frequency \( \equiv \omega_m = \left( \omega_1 - \omega_2 \right) / 2 \)

Modulation propagation number \( \equiv k_m = \left( k_1 - k_2 \right) / 2 \)

Time-varying modulation amplitude \( \equiv E_0(x,t) = 2E_0 \cos \left[ k_m x - \omega_m t \right] \)

Superimposed wavefunction: \( \bar{E} = E_0(x,t) \cos \left[ kx - \bar{\omega}t \right] \)

For large \( \omega \) if \( \omega_1 \approx \omega_2 \) then \( \omega >> \omega_m \) we will have a slowly varying amplitude with a rapidly oscillating wave.
Irradiance of two superimposed waves with different frequencies

\[ E_0^2(x,t) = 4 E_{01}^2 \cos^2 \left[ k_m x - \omega_m t \right] = 2 E_{01}^2 \left[ 1 + \cos \left( 2k_m x - 2\omega_m t \right) \right] \]

Beat frequency \(2\omega_m = \omega_1 - \omega_2\) or oscillation frequency of the \(E_0^2(x,t)\)

Amplitude, \(E_0\), oscillates at \(\omega_m\), the modulation frequency

Irradiance, \(E_0^2\), varies at \(2\omega_m\), twice the modulation frequency

Two waves with different amplitudes produce beats with less contrast.
Beats

Superposition of two waves

Irradiance $2\omega_m$ \[ \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \]

Amplitude $\omega_m$ \[ 2E_{01} \]

Distance \[ x \times 10^{-6} \]
Phase velocity of a wave

Constant-phase point: a point on a progressive wave with a constant magnitude of the disturbance.

\[ \psi(x, t) = A \sin(kx \mp \omega t + \varepsilon) \]

Phase velocity of a wave is speed of the motion of a constant-phase point on a disturbance

\[ V_{phase} = \left. \frac{\partial x}{\partial t} \right|_\phi \]

Phase velocity is speed of the motion of the disturbance.

Ex: calculate value of the phase velocity for the above wave.

Phase of a harmonic wave at time \( t \): \( \phi = kx \mp \omega t + \varepsilon \)

For constant phase: \[ \frac{\partial \phi}{\partial t} = 0 \rightarrow k \left. \frac{\partial x}{\partial t} \right|_\phi \mp \omega = 0 \rightarrow V_{phase} = \pm \frac{\omega}{k} \]
Group velocity

In nondispersive media velocity of a wave is independent of its frequency.

For a single frequency wave there is one velocity and that is

\[ V_{\text{phase}} = \frac{\omega}{k} \]

When a wave is composed of different frequency elements, the resulting disturbance will travel with different velocity than phase velocity of its components.

\[ E = 2E_{01} \cos \left[ k_m x - \omega_m t \right] \times \cos \left[ k x - \omega t \right] \]

\[ V_{\text{phase}} = \frac{\omega}{k} \text{ velocity of a constant phase point on the high frequency wave} \]

\[ V_{\text{group}} = \frac{\omega_m}{k_m} = \left( \frac{d\omega}{dk} \right)_{\omega} \text{ velocity of a constant amplitude point on the modulation envelope} \]

\( V_g \) may be smaller, equal, or larger than \( V_p \)

To calculate \( V_p \) and \( V_g \) we need the dispersion relation \( \omega = \omega(k) \)
Dispersion relation $\omega$ v.s. $k$ or $\omega = f(k)$

- **Phase velocity** for a given frequency is the slope of a line on the dispersion curve that connects that point to the origin or $w/k$.

- **Group velocity** for that frequency is the slope of the dispersion curve at that point or $dw/dk$.

- We also may have a gap in the dispersion relation for a frequency band. In that case the velocities are **not defined** because waves cannot propagate.

\[
V_g = V_p \quad \text{or} \quad \frac{d\omega}{dk} = \frac{\omega}{k}
\]

\[
V_g < V_p \quad \text{or} \quad \frac{d\omega}{dk} < \frac{\omega}{k}
\]

\[
V_g > V_p \quad \text{or} \quad \frac{d\omega}{dk} > \frac{\omega}{k}
\]
Finite waves

- Finite wave: any wave starts and ends in a certain time interval
- Any finite wave can be viewed as a really long pulse
- Any pulse is a result of superposition of numerous different frequency harmonic waves called Fourier components.
- Wave packet is a localized pulse that is composed of many waves that cancel each other everywhere else but at a certain interval in space.
- We need to study Fourier Analysis to understand actual waves, pulses, and wave packets.
- Width of a wave packet is proportional to the range of $k_m$ of the wave packet.
- Since each component of the wave packet has different phase velocity in the medium, through the relationship $V_p = \omega/k$, $k$ of the components change in the dispersive media.
- As a result $k_m$ of the modulation disturbance changes
- Consequently group velocity changes.
- This results in change of the width of the wave packet.
- So wave packets inside a medium may spread or become narrower.