Laser Physics 168
L2
Interaction of Radiation with Atoms and Ions
Black body radiation

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In this chapter

1. Introduction
2. Summary of Blackbody Radiation Theory
3. Spontaneous Emission
4. Absorption and Stimulated Emission
5. Lineshape
6. Amplification
7. Broadening Mechanisms

Definition of the parameters

ρ : enrgy density inside the cavity $\left(Joules / L^3\right)$

ε_0 : dielectric constant of the vacuum $\left(8.85\times10^{-12} F / m\right)$

μ_0: magnetic permeability of vacuum $\left(4\pi\times10^{-7} H / m\right)$

ρ_v : energy per unit volume $\left(density\right)$ at frequency v $\left(Joules / L^3\right)$

$\rho(v)$: energy per unit volum per unit frequency at $v\left(Joules.T / L^3\right)$

$p(v)$: mode density per unit frequency at v $\left(1 / L^3.energy\right)$

N_v : number of modes in a volume V bettween frequencies 0 and v

c_n : c / n speed of light in the media (L / T)

m,p,q : positive integers

a,b,d : cavity dimensions

$\langle\varepsilon\rangle$: avarage energy per mode

$g(v)$: the lineshape function (T a probability function)

$\sigma(v)$: stimulated emission cross section $\left(L^2\right)$

Introduction

- Focus on interaction of radiation with atoms and ions that are weakly interacting with their surrounding.
- Gas phase or impurity ions in an ionic crystal (or doped ions)
- Simplifying assumptions: Limit the interaction radiation with only
  - active material (not the host)
  - dilute medium (no interaction between the dopants)
  - low intensity radiation (linear optics)

Summary of Blackbody Radiation Theory

Blackbody radiation is very important for understanding of radiation in general and lasers in particular.

For a cavity at thermal equilibrium rate of emission and absorption of radiation has to be equal.

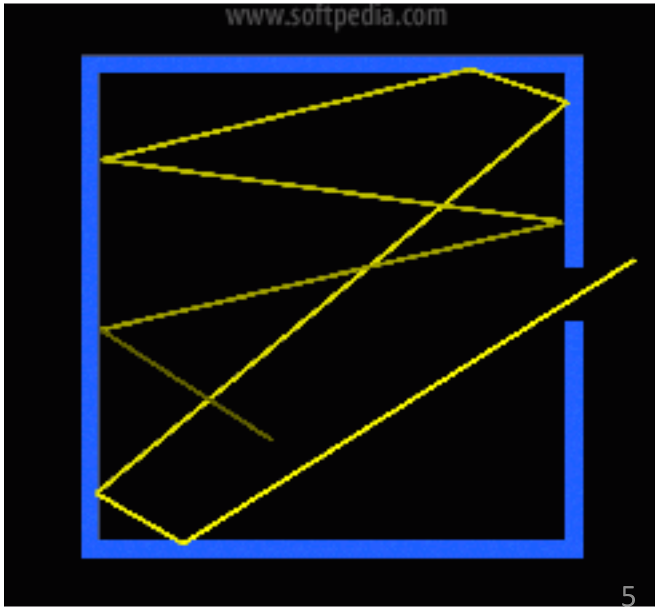
Energy dinsity inside the cavity:

$$\rho = \left\langle \frac{1}{2}\varepsilon E^2 \right\rangle_t + \left\langle \frac{1}{2}\mu H^2 \right\rangle_t$$

ε : dielectric constant

μ : magnetic permeability

The time average is over a cycle of radiation field.



Modes of a rectangular cavity

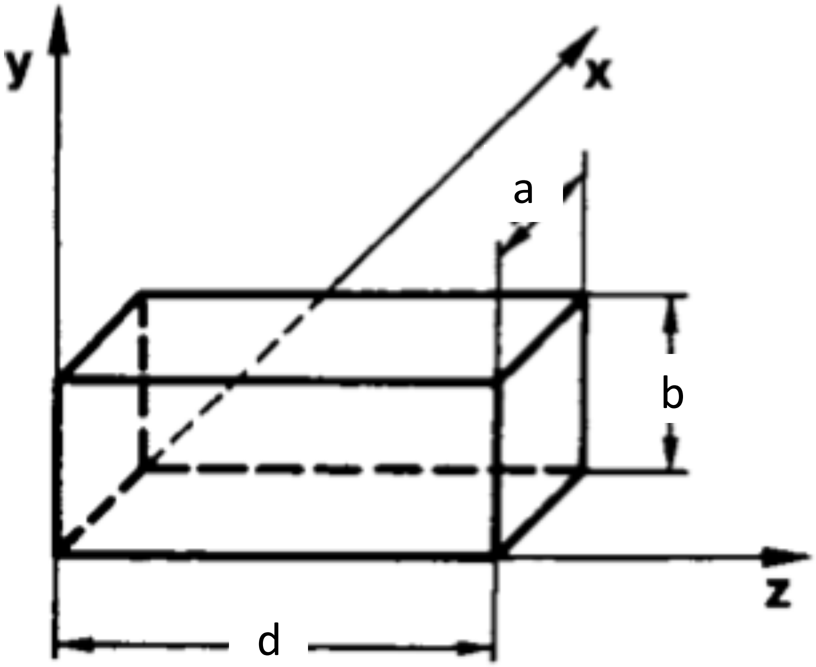
Starting from the the EM (Maxwell) wave equation

$$\nabla^2\mathbf{E} - \frac{1}{c_n^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

and proper boundary condtions for a conducting rectangular cavity $(\mathbf{E}\times\mathbf{n} = 0)$ one can show that $\rho(\nu)$ the **number of cavity oscillation modes per unit volume and per unit frequency range** is:

$$p(\nu) = \frac{1}{V}\frac{d\mathrm{N}}{d\nu} = \frac{8\pi\nu^2}{c_n^3}$$

Rectangular cavity with perfectly conducting walls at temperature T



(Svelto)

Modes of a rectangular cavity

The E-field must satisfy the wave equation: $\nabla^2\mathbf{E}-\frac{1}{c_n^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0$

Boundary condtion for the E-field at each wall: $\mathbf{E}\times\mathbf{n}=0$

where **n** is the unit vector perpendicular to the wall. This means **tangantial components of the E at the walls of a conductor vanish**.

Assume $\mathbf{E}(x,y,z,t)=\text{space part}\times\text{time part}=\mathbf{u}(x,y,z)E(t)$

$$\left.\begin{aligned}&\nabla^2\mathbf{u}=-k^2\mathbf{u}\\&\frac{d^2E}{dt^2}=-\left(c_nk^2\right)E\end{aligned}\right\}\quad\begin{aligned}&\leftarrow\mathbf{u}(x,y,z)\text{ the Helmholtz equation}\\&\leftarrow E(t)=E_0\cos(\omega t+\phi)\text{ where }\omega=c_nk\end{aligned}$$

$$\mathbf{E}(x,y,z,t)=\mathbf{u}(x,y,z)E_0\cos(\omega t+\phi)=E_0\mathbf{u}(x,y,z)e^{(j\omega t+\phi)}$$

This soution corresponds to standing wave inside the cavity.

Amplitude of the oscillations at a given point of cavity is constant.

These solutions are known as **EM modes of the cavity**.

Meaning of a mode

- Possible solutions of the wave equation
- Think about modes of vibrations of a taut string
- Energy of the modes will vary the higher the frequency the higher the energy
- Number of modes: number of solutions

Modes of a rectangular cavity

Solving the Helmholtz equation $\nabla^2 \mathbf{u} = -k^2 \mathbf{u}$ with the BC $\mathbf{E} \times \mathbf{n} = 0$

with $\mathbf{u} = u_x i + u_y j + u_z$ the Laplacian separates $\nabla^2 u_i = \dfrac{\partial^2 u_i}{\partial x^2} + \dfrac{\partial^2 u_i}{\partial y^2} + \dfrac{\partial^2 u_i}{\partial z^2}$

$$\left.\begin{aligned} u_x &= e_x \cos k_x x \sin k_y y \sin k_z z \\ u_y &= e_y \sin k_x x \cos k_y y \sin k_z z \\ u_z &= e_z \sin k_x x \sin k_y y \cos k_z z \end{aligned}\right\} \text{A solution that satisfiels the BC at } x, y, z = 0$$

Condition for the solution to satisfy the Helmholtz eq. for any e_x, e_y, e_z :

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

with this condition the u$_i$ solutions should also satisfy the BC on the other side of the walls $x = a,\ y = b,\ z =$ d. We get:

$$k_x = \frac{m\pi}{a}, k_y = \frac{p\pi}{b}, k_z = \frac{q\pi}{d} \text{ where } m, p, q = 1, 2, 3, ...$$

m, p, q represent the **number of nodes that the standing wave has along the each direction of** $x,\ y,\ z$

Modes of a rectangular cavity

Angular frequency of a mode $\omega = c_n k$ is determined by l, m, n

$$\omega_{mpq} = c_n k = c_n \left(k_x^2 + k_y^2 + k_z^2\right)^{1/2} = c_n \left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{p\pi}{b}\right)^2 + \left(\frac{q\pi}{d}\right)^2\right)^{1/2}$$

But the mode will be completely determined once e$_x, e_y, e_z$ are determined.

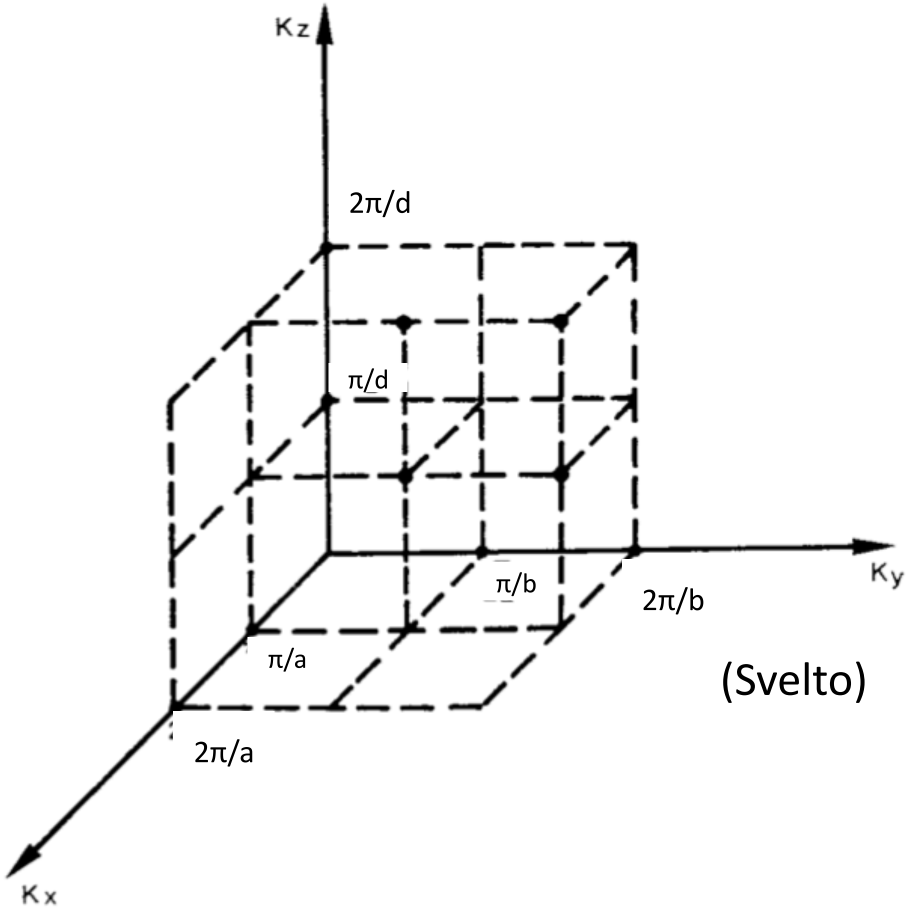
That is done by another ondition imposed by Maxwell equations.

$$\left.\begin{aligned} &\nabla \cdot \mathbf{u} = \mathbf{0} \rightarrow \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \\ &u_x = e_x \cos k_x x \sin k_y y \sin k_z z \\ &u_y = e_y \sin k_x x \cos k_y y \sin k_z z \\ &u_z = e_z \sin k_x x \sin k_y y \cos k_z z \end{aligned}\right\} \rightarrow \mathbf{e} \cdot \mathbf{k} = 0 \text{ where} \left\{\begin{aligned} &\mathbf{e} = e_x i + e_y j + e_z k \\ &\mathbf{k} = k_x i + k_y j + k_z k \\ &\text{and } \mathbf{e} \perp \mathbf{k} \end{aligned}\right.$$

once we fix m, p, q $\left(\text{or } k\right)$ then the fact that **e** has to be perpendicular to **k** fixes one of the components of the **e** so only two independet modes can exist on the plane perpendicular to the **k**.

Number of resonant modes below a certain frequency

- What is density of modes in the cavity
- Each point of the lattice corresponds tc two modes



(Svelto)

Number of resonant modes below a certain frequency

N_ν : number of resonant modes with frequency between 0 and ν

N_ν : number of modes whose wavevetor $\mathbf{k}$ is between 0 and $2\pi\nu / c_n$

possible values given for $\mathbf{k}$: vectors connecting the origin to the nodal points of the 3D lattice shown.

Only the positve octant counts: 1/8

$$N_\nu = \text{modes per k}\,\frac{\frac{1}{8}\text{volume of the sphere centered at origin and radius k}}{\text{volume of the unit cell }\left(\frac{\pi}{a},\frac{\pi}{b},\frac{\pi}{d}\right)}$$

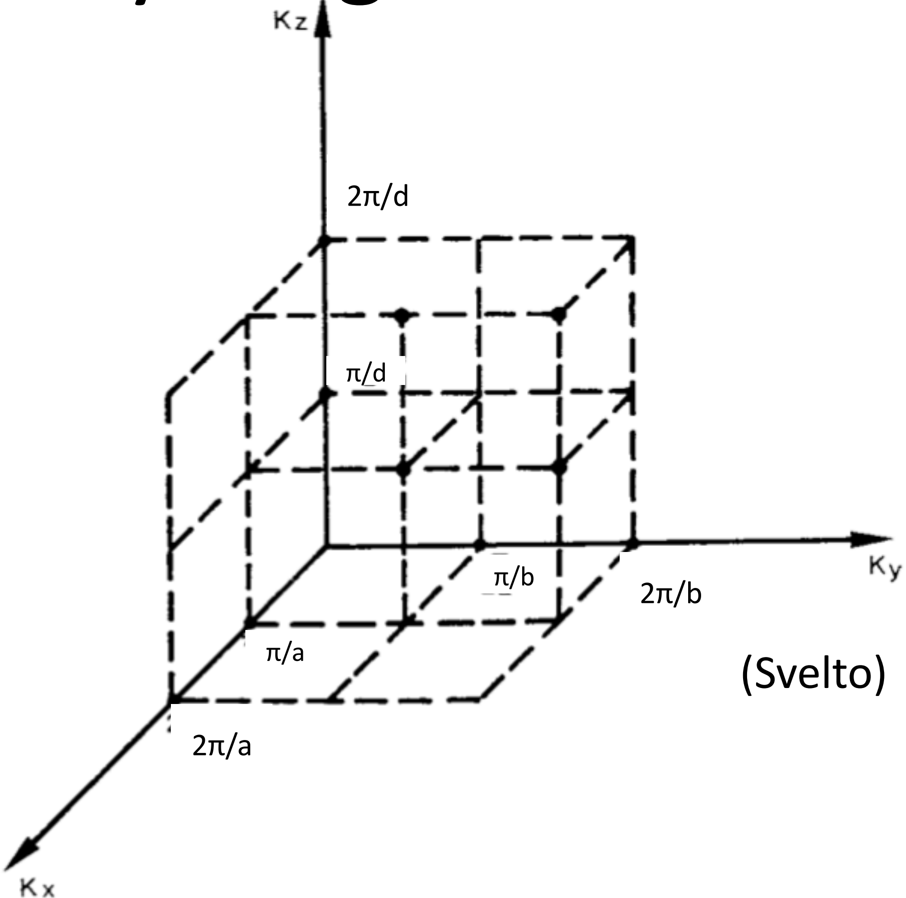
$$N_\nu = 2\,\frac{\frac{1}{8}\frac{4}{3}\pi\left(\frac{2\pi\nu}{c_n}\right)^3}{\frac{\pi}{a}\frac{\pi}{b}\frac{\pi}{d}}$$

Number of modes per unit volume per unit frequency range

$$N_\nu = 2\frac{\frac{1}{8}\frac{4}{3}\pi\left(\frac{2\pi\nu}{c_n}\right)^3}{\frac{\pi^3}{abd}}$$

$\rho(\nu)$: number of modes per **unit volume** and per **unit frequency range**.

$$\rho(\nu) = \frac{1}{V}\frac{dN}{d\nu} = \frac{8\pi\nu^2}{c_n^3}$$



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note: do not worry about n and n_g on page 177 of Verdeyene at this point. Assume there is only one index of refraction and that is *n*.

Average energy contained in a mode

Assume the cavity walls are kept at constant temperature T.

According to Boltzman's statistics probability that the energy of a given mode lies between E and $E+dE$ is:

$$dp = Ce^{-E/k_BT}dE$$

C is a constant that can be calculateed from $\int_0^\infty Ce^{-E/k_BT}\,dE \ = 1$

$$\langle E\rangle = \frac{\int_0^\infty E\,dp}{\int_0^\infty dp} = \frac{\int_0^\infty Ee^{-E/k_BT}\,dE}{\int_0^\infty e^{-E/k_BT}\,dE} = k_BT$$

Note : average energy content of a mode $\langle E\rangle$ **only depends on T**.

Exercise : calculate $\langle E\rangle$ and C

Modes of a rectangular cavity and the Rayleigh-Jeans radiation formula

Calculate the **spectral energy distribution** using Boltzman statistics

$$\rho_\nu = p(\nu)\langle E\rangle$$

$p(\nu)$: number of modes per unit volume per unit frequency range

$\langle E\rangle$: average energy contained in each mode (we need to find this)

$$\left.\begin{aligned} p(\nu) &= \frac{1}{V}\frac{d\mathrm{N}}{d\nu} = \frac{8\pi\nu^2}{c_n^3} \\ \langle E\rangle &= k_B T \end{aligned}\right\}$$

$\rho_\nu = \left(\dfrac{8\pi\nu^2}{c_n^3}\right)k_B T$ ← The **Rayleigh - Jeans** radiation formula

What is wrong with it?

Planck’s hypothesis

The energy in a given mode of a cavity could not have any arbitrary value between 0 and ∞. The allowed values of this energy should be **integral multiples of a fundamental quantity**, **proportional to the frequency of the mode**.

In other words energy of a mode could be written as:

$$\varepsilon_n = nh\nu \qquad n = 1, 2, 3, ...$$

where n is a positive integer and h is the Planck’s constant.

$$h = 6.63 \times 10^{-34}\, j.s$$

This implies that energy exchange between the cavity and its walls must involve a discrete amount of energy $h\nu$

Average energy of a cavity mode using Planck's hypothesis

$$\langle \varepsilon \rangle = \frac{\sum_0^\infty nh\nu e^{-nh\nu/k_BT}}{\sum_0^\infty e^{-nh\nu/k_BT}} = \frac{h\nu}{e^{h\nu/k_BT}-1}$$

For $h\nu \ll k_BT$, $\langle \varepsilon \rangle \simeq \dfrac{h\nu}{1+\dfrac{h\nu}{k_BT}-1} = k_BT$

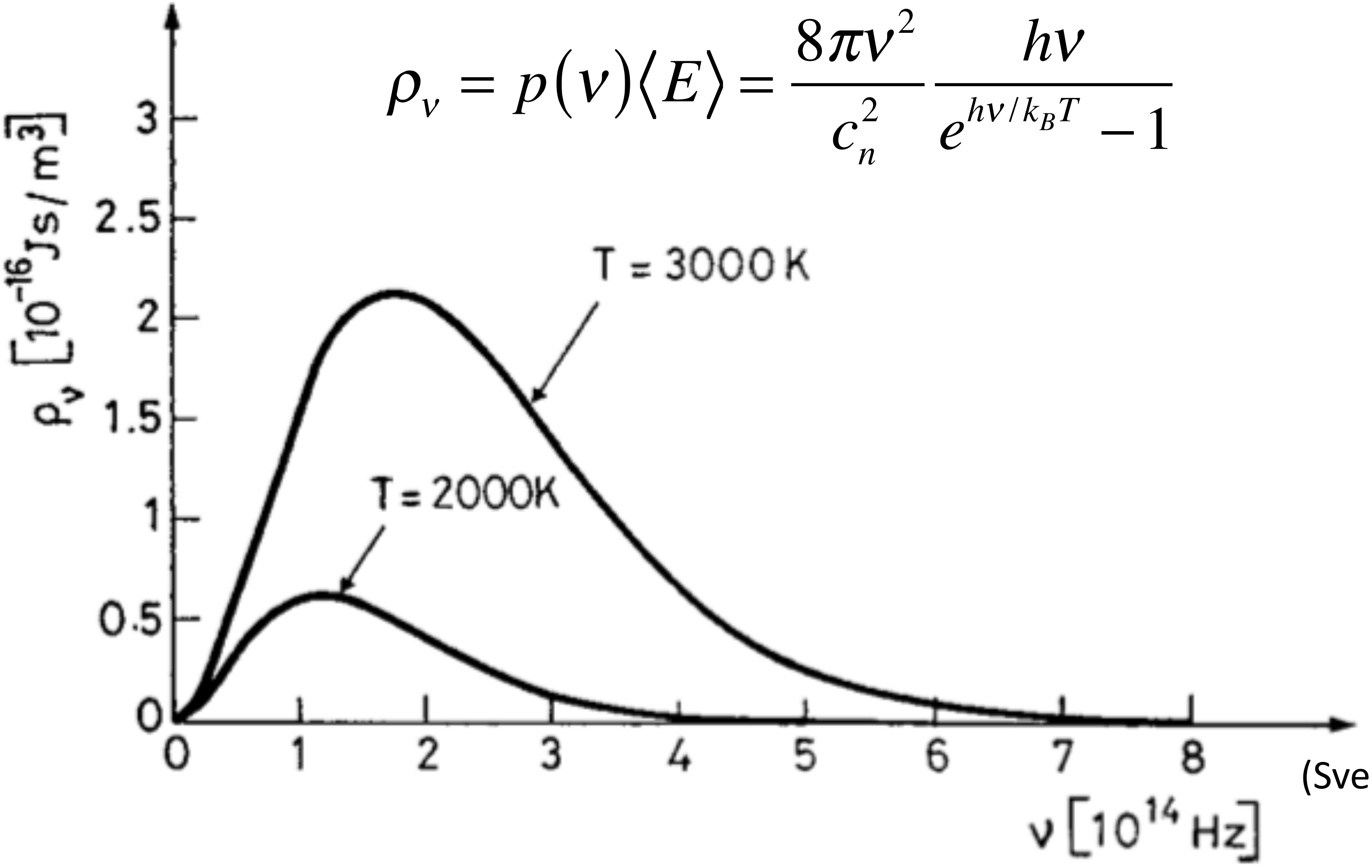
compatible with the classical limit at low frequency range.

Planck's formula for radiation energy density:

$$\rho_\nu = p(\nu)\langle \varepsilon \rangle = \frac{8\pi\nu^2}{c_n^3}\frac{h\nu}{e^{h\nu/k_BT}-1}$$

in agreement with experimental results. *What* if $\nu \to \infty$?

Spectral energy distribution vs. frequency for two different Ts



$$\rho_\nu = p(\nu)\langle E\rangle = \frac{8\pi\nu^2}{c_n^2}\frac{h\nu}{e^{h\nu/k_BT}-1}$$

(Svelto)

Uncertainty principle, harmonic oscillator

For a mechanical oscilator total energy is

$$E = \frac{1}{2}kq_x^2 + \frac{1}{2}\frac{p_x^2}{m}$$

oscillation occurs because potential energy, is periodically transformed into kinetic energy.

Energy of a given oscillator mode is quantized

$$E = \frac{1}{2}h\nu + nh\nu$$

zero point energy can not be zero because that requires both p_x and q_x energy be zero which is not allowed by uncertainty principle.

The canonical pair that cannot be measured simultaneously with arbitrary precision

$$\Delta p_x \Delta q_x \geq \hbar / 2$$

Planck's Hypothesis and Field Quantization

1900 Planck's hypothesis quantization of EM radiation $E = nh\nu$

1904 Einstein used it to explain the photoelectric effect

1927 Dirac's complete justification of quantization of EM radiation by Quantum Field Theory

A mode of the cavity, characterized by a **standing EM wave pattern**, and its resonance frequency of ν.

Energy density of the mode: $\rho = \left\langle \frac{1}{2}\varepsilon E^2 \right\rangle_t + \left\langle \frac{1}{2}\mu H^2 \right\rangle_t$

Energy of the mode: $E = \int \rho \, dV \quad \left(V \text{ volume of the cavity}\right)$

$E_x\left(\mathbf{r},t\right)$ & $H_y\left(\mathbf{r},t\right)$: transverse components of the E & M fields

Uncertainty principle, harmonic oscillator & quantization of the EM radiation

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The canonical pair that cannot be measured simultaneously with arbitrary precision

$$\Delta p_x \Delta q_x \geq \hbar / 2$$

In the electromagnetic oscillator represented by the cavity mode, oscillation occures because the energy is periodically transformed between electric and magnetic fields.

$$E = \int \frac{1}{2} \varepsilon \left\langle E_x^2 \right\rangle dV + \int \frac{1}{2} \mu \left\langle H_y^2 \right\rangle dV$$

Energy of a given cavity mode is quantized

$$\mathrm{E} = \frac{1}{2} h\nu + nh\nu$$

zero point energy can not be zero because that requires both E and M energy be zero which is not allowed by uncertainty principle.

The canonical pair that cannot be measured simultaneously with arbitrary precision are $\mathrm{E}_x(\mathbf{r}, t)$ and $\mathrm{H}_y(\mathbf{r}, t)$

Energy levels of a cavity mode

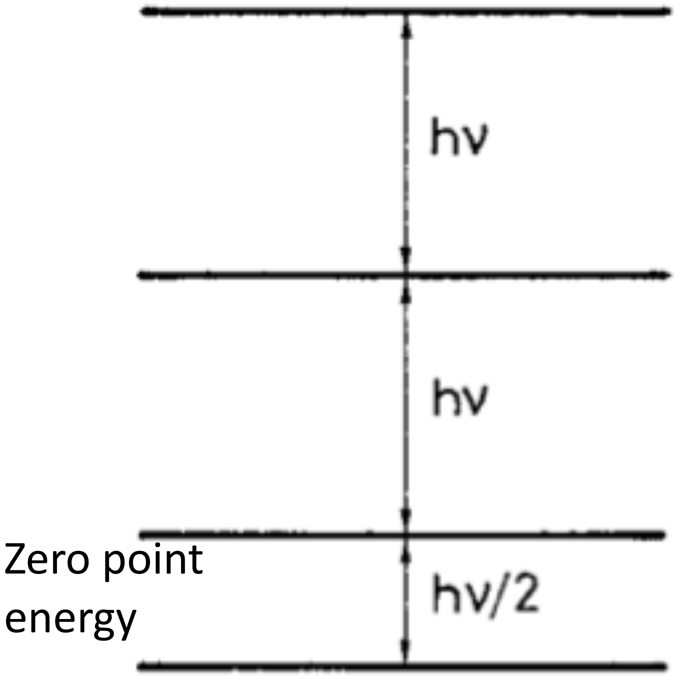
$$E = \frac{1}{2}h\nu + nh\nu$$

$$\omega = 2\pi\nu$$

$$\hbar = \frac{h}{2\pi}$$

$$E = \frac{1}{2}\hbar\omega + n\hbar\omega$$

$$n = 0,1,2,3,...$$



(Svelto)

Average number of optical photons per mode at room temperature in a cavity

$\langle \phi \rangle$, the avaeage number of photons for each mode is:

$$\langle \phi \rangle = \frac{E \text{ of a mode}}{E \text{ of a photon}} = \frac{\langle E \rangle}{h\nu} = \frac{1}{e^{h\nu/k_B T} - 1}$$

$$\left.\begin{array}{l} \nu \approx 4 \times 10^{14}\, Hz \text{ (optical frequencies)} \\ h = 6.632 \times 10^{-34}\, j.s \end{array}\right\} h\nu \approx 1eV$$

$$\left.\begin{array}{l} T = 300K \text{(room temperature)} \\ k_B = 8.6173324(78) \times 10^{-5}\, eV / K \end{array}\right\} kT = 0.025eV$$

Conclusion: hν>>k$_B$T(energy of an atom at T)

Energy needed to excite a mode in optical freq.s>>kinetic energy of an atom at room temperature

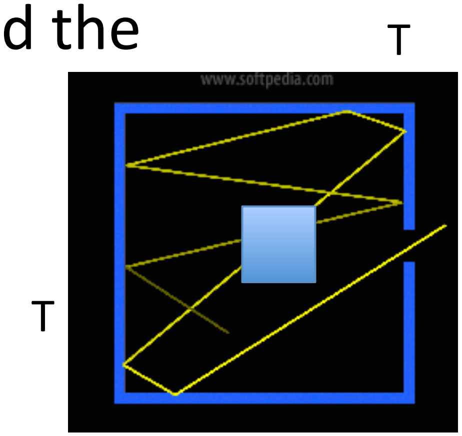
$\langle \phi \rangle \simeq e^{-h\nu/k_B T} \simeq e^{-40} \simeq 0$ no chance of laser actioon at room T

Spontaneous Emission: light from the sun or from any ordinary lamps

- Semiclassical approach (fails to describe the phenomenon of spontaneous emission in a correct way)
  - atoms are treated as quantized (i.e. treated according to quantum mechanics)
  - the fields are treated classically (i.e. treated through Maxwell's equations).
- Full quantum theory (correctly describe the phenomenon)
  - Atoms are quantized by quantum mechanics and
  - The fields are quantized by quantum field theory.
- Einstein thermodynamic treatment (we will cover this one)

Einstein thermodynamics treatment

- Properly and elegantly explains both spontaneous and stimulated transitions
- Material is placed in a blackbody cavity with its walls at constant temperature T
- At thermal equilibrium the EM energy density with a spectral distribution (ρ_ν) will be established and the material is immersed in this radiation.



- The material will experience
  - absorption
  - stimulated emission
  - spontaneous emission.
- Since there is thermal equilibrium number of transitions from level 2 to 1 is equal to the number of transitions from level 1 to 2

Probability of Spontaneous Emission

N : number of atoms or molecules per unit volume at time t that are at a given energy level.

N_1 & N_2 : population of levels 1 and 2 at t

Probability of spontaneous emission:

$$\left(\frac{dN_2}{dt}\right)_{sp} = -A_{21}N_2 = -\frac{\mathrm{N}_2}{\tau_r} \quad \text{and} \quad \left(\frac{dN_2}{dt}\right)_{nr} = -\frac{N_2}{\tau_{nr}}$$

A_{21} : the Einstein A coefficient (positive) obtained by thermodynnamics considerations

$\tau_{sp} = 1/A_{21}$: the spontaneous emission (radiative lifetime)

The numerical values depend on the particular transition

τ_{nr} : is the non-radiative lifetime

Probability of stimulated emission and absorption

Probability of stimulatd emission and absorption:

$$\left(\frac{dN_2}{dt}\right)_{st} = -B_{21}N_2\rho(\nu) = -\left(\frac{dN_1}{dt}\right)_{st}$$

$$\left(\frac{dN_2}{dt}\right)_{abs} = +B_{12}N_1\rho(\nu) = -\left(\frac{dN_1}{dt}\right)_{abs}$$

B_{21} : the rate of stimulated emission from $2 \rightarrow 1$.

B_{12} : the rate of (stimulated) absorption from $1 \rightarrow 2$

The numerical values of Bs **depend on the particular transition** and photon flux (photones per area).

Majic of the stimulated emission:

The new photon has the **same frequency, polarization, direction, and phase** as the stimulating photon.

Probability of transitions between degenerate and non-degenerate states

Probability of stimulatd emission and absorption:

$$\left(\frac{dN_2}{dt}\right)_{st} = -B_{21}N_2 = -\sigma_{21}F \quad \text{and} \quad \left(\frac{dN_1}{dt}\right)_{a} = -B_{12}N_1 = -\sigma_{12}F$$

It was shown by Enistein at early 20th century that :

For trasition between **non - degenerate** states: $B_{21} = B_{12}$

For trasition between **degenerate** states: $g_2 B_{21} = g_1 B_{12}$

g_1 and g_2 : degeneracy of levels 1 and 2.

For equal photon flux F: $\mathrm{g}_2\sigma_{21} = g_1\sigma_{12}$

σ_{21} and σ_{12} are the emission and absorption cross sections.

In sammary we can say each stimulated emission process **creates** a photon and each absorption process **anihilates** a photon

Absorption and Stimulated Emission

At thermodynamic equilibrium each emission must be balanced by an absorption process.

N_1 & N_2 equilibrium populations of levels 1 and 2

$$\left.\frac{dN_2}{dt}\right|_{radiative} = -A_{21}N_2 + B_{12}N_1\rho(\nu) - B_{21}N_2\rho(\nu) = -\left.\frac{dN_1}{dt}\right|_{rdiative} = 0$$

$$\left.\begin{aligned} & A_{21}N_2 + B_{21}N_2\rho(\nu) = B_{12}N_1\rho(\nu) \\ & \text{Using Boltzman stat. } \frac{N_2}{N_1} = \frac{g_2}{g_1}e^{-h\nu/k_BT} \end{aligned}\right\} \Rightarrow \frac{g_2}{g_1}e^{-h\nu/k_BT} = \frac{B_{12}\rho(\nu)}{A_{21} + B_{21}\rho(\nu)}$$

g_1(or g_2) are the number of ways an atom can have the energy E_1(or E_2)

For a simple atom $g = 2J+1$ whre J is the total angular momentum quantum number.

Solve for $\rho(\nu)$: $\rho(\nu) = \dfrac{A_{21}\left(g_2/g_1\right)e^{-h\nu/k_BT}}{B_{12} - B_{21}\left(g_2/g_1\right)e^{-h\nu/k_BT}}$

Absorption and Stimulated Emission

$$\underbrace{\rho(\nu)=\frac{A_{21}}{B_{21}}\frac{1}{\frac{B_{12}g_1}{B_{21}g_2}e^{h\nu/k_BT}-1}}_{\text{Einstein's radiation formula at thermal equilibrium}} \quad \& \quad \underbrace{\rho(\nu)=\frac{8\pi\nu^2}{c_n^3}\frac{h\nu}{e^{h\nu/k_BT}-1}}_{\text{Planck's BBR formula}}$$

To make this to look like Planck's fromula, Einstein required:

the **probability of absorption and SE for the atoms in the cavity**

due to BBR are equal. $\left\{\begin{aligned} B_{12}g_1 &= B_{21}g_2 \\ \frac{A_{21}}{B_{21}} &= \frac{8\pi n^3 h\nu_0^2}{c^3} \end{aligned}\right\}$ **Einstein's conditions**

These conditions show that all three processes are related.

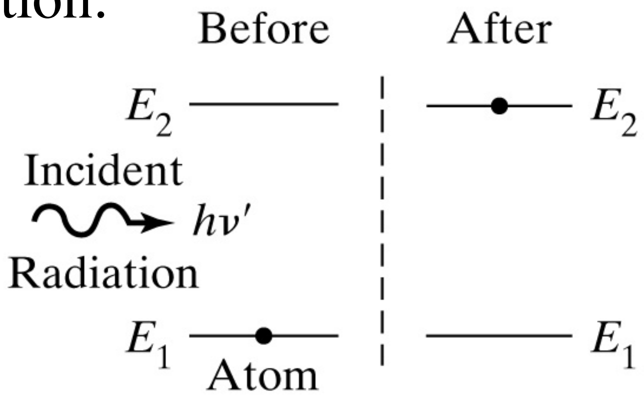
We have to find A & Bs to complete the solution of the problem.

Rate of transitions depend on the atomic structure but other factors such as collisions, lattice vibrations, etc. come to play.

Atomic and molecular line shapes

So far we have assumed atomic lines are infinitely sharp, energy levels are exactly defined and the emitted or absorbed photon has a unique frequency defined by the difference between the energy levels involved in the transition:

$$\nu_0 = \frac{E_2 - E_1}{h}$$



(a) Stimulated absorption

(Svelto)

This is in conflict with uncertainty principle.

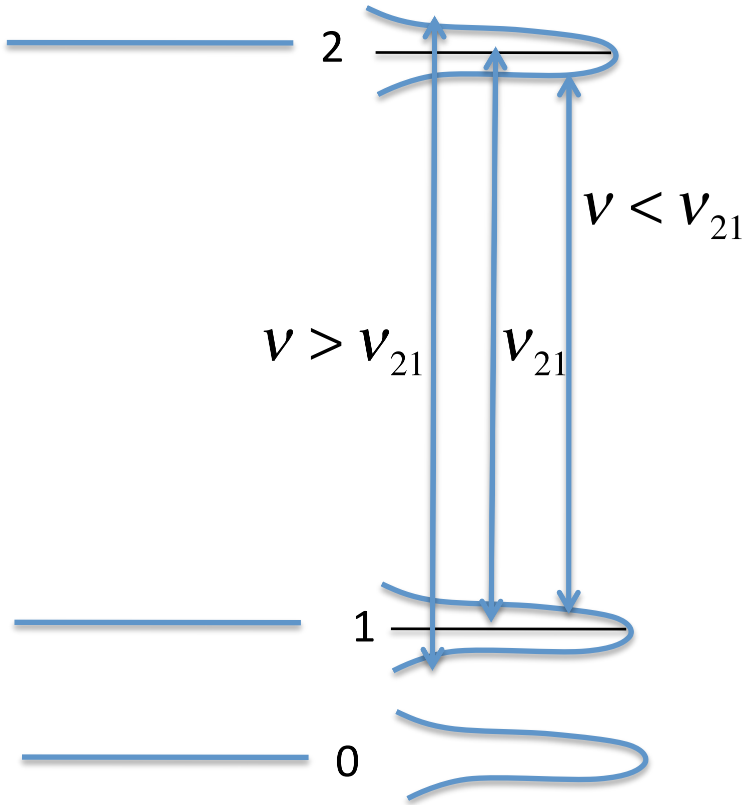
There has to be a finite linewidth.

In an unsumble of atoms there are also many other reasons for the atomic lines to be broader.

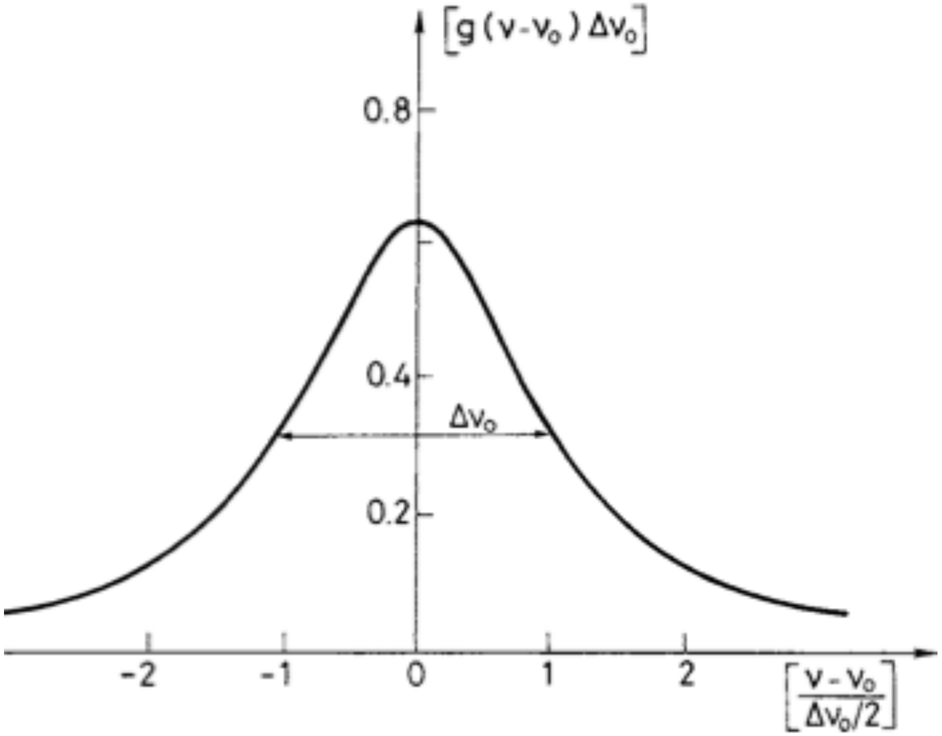
BBR has a very broad emission spectrum.

The systems we deal with will usually have narrow line widths $\Delta\nu << \nu_0$, where ν_0 is the central frequency.

Evolution of the energy level diagram



Zero width lines Broadened levels



(Svelto)

Spectral line shape
Example: a Lorentzian

Definition of g(*ν*) the lineshape function

$g(\nu')d\nu' =$ probability that a spontaneously emitted photon will appear at a frequency between ν' and $\nu' + d\nu'$

$\int_0^\infty g(\nu')d\nu' = 1$ (photon has to have some frequency)

$g(\nu_0) \simeq 1/\Delta\nu$

$\nu_0 = \nu_{21}$ is frequency of the peak

$\Delta\nu$ is the FWHM of the emission spectra

Broadeneing of both upper and lower states contribute to $g(\nu)$

Spectral distribution of the emitted power :

$\{I(\nu)d\nu\}\{\text{surface area}\} = \{h\nu A_{21}N_2 g(\nu)d\nu\}\{\text{volume of surface}\}$

we can measure $g(\nu)d\nu$ with very narrow band-pass instruments.

Fabry-Perot cavities are good for this and will cover them later.

Other Definitions of g(ν)

$g(\nu')d\nu' =$ relative strength of absorption of radiation in the inteval of ν' and $\nu' + d\nu'$ by atoms in state 1

$g(\nu')d\nu' =$ relative strength of stimulation by radiation in the inteval of ν' and $\nu' + d\nu'$ in inducing the atoms in state 2 to release their internal energy

Now we need to modify the rate quations to account for the lineshape

$$\left.\frac{dN_2}{dt}\right|_{radiative} = -A_{21}N_2 + B_{12}N_1\rho(\nu) - B_{21}N_2\rho(\nu)$$

$$\left.\frac{dN_2}{dt}\right|_{radiative} = -A_{21}N_2\int_0^\infty g(\nu')d\nu' + B_{12}N_1\int_0^\infty \rho(\nu')g(\nu')d\nu'$$

$$-B_{21}N_2\int_0^\infty \rho(\nu')g(\nu')d\nu'$$

Photons in the cavity and line shape

$$\left.\frac{dN_2}{dt}\right|_{radiative} = -A_{21}N_2\int_0^\infty g(\nu')d\nu' + B_{12}N_1\int_0^\infty \rho(\nu')g(\nu')d\nu'$$

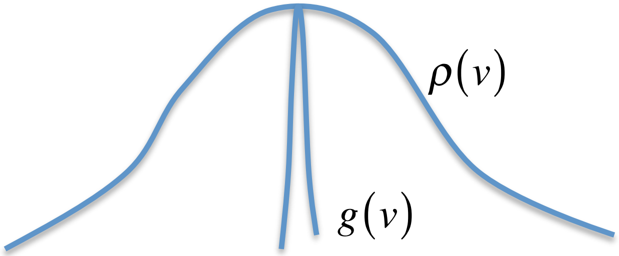
$$-B_{21}N_2\int_0^\infty \rho(\nu')g(\nu')d\nu'$$

$\rho(\nu')$ is the radiation hitting the atoms (pump or cavity feedback)

Some special cases:

1) If $g(\nu')$ **is much narrower than** $\rho(\nu')$ or radiation hitting the atoms is very broad we can evaluate $\rho(\nu')$ at $\nu'=\nu$ and pull it out of the integral and with $\int_0^\infty g(\nu')d\nu' = 1$ we get the original formula for dN_2/dt :

$$\left.\frac{dN_2}{dt}\right|_{radiative} = -A_{21}N_2 + B_{12}N_1\rho(\nu) - B_{21}N_2\rho(\nu)$$

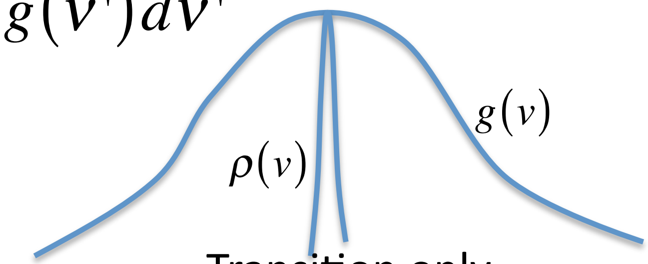


Photons in the cavity and line shape

2) **Other more common case in lasers** : $\rho(\nu)$ is much narrower than $g(\nu)$ then we assume all of the photons have single frequency (pumping with a laser or selecting the cavity with a sharp frequency band.

$\rho(\nu') \approx \rho_\nu \delta(\nu' - \nu)$ then $\int_0^\infty \rho_\nu \delta(\nu' - \nu) g(\nu') d\nu' = \rho_\nu g(\nu)$

$$\left.\frac{dN_2}{dt}\right|_{radiative} = -A_{21}N_2 \int_0^\infty g(\nu')d\nu' + B_{12}N_1 \int_0^\infty \rho(\nu')g(\nu')d\nu' - B_{21}N_2 \int_0^\infty \rho(\nu')g(\nu')d\nu'$$



Transition only happens here

$$\left.\frac{dN_2}{dt}\right|_{radiative} = -A_{21}N_2 - B_{21}N_2\rho_\nu g(\nu) + B_{12}N_1\rho_\nu g(\nu)$$

for the case of very narrow $\rho(\nu)$. We will use the Einstein conditions to simplify this rate and define the stimulated emission cross section.

Stimulated emission cross section

We express the radiation density as a function of intentity of radiation:

Recall from E&M: $\rho_v\left(\frac{J}{L^3}\right)=\frac{I_v}{c_n}\left(\frac{(J/T)/L^2}{L/T}\right)\rightarrow\rho_v=\frac{I_v}{c_n}$

Einstein's conditions: $B_{12}=\frac{g_2}{g_1}B_{21}$ & $B_{21}=\frac{c^3}{8\pi n^3 v^2}\frac{A_{21}}{h\nu}=\frac{c\lambda_0^2}{8\pi n^3}\frac{A_{21}}{h\nu}$

$$\left.\frac{dN_2}{dt}\right|_{radiative}=-A_{21}N_2-\frac{c\lambda_0^2}{8\pi n^3}\frac{A_{21}}{h\nu}N_2\rho_v g(\nu)+\frac{g_1}{g_2}\frac{c\lambda_0^2}{8\pi n^3}\frac{A_{21}}{h\nu}N_1\rho_v g(\nu)$$

$$\left.\frac{dN_2}{dt}\right|_{radiative}=-A_{21}N_2-\left\{A_{21}\frac{\lambda_0^2}{8\pi n^2}g(\nu)\right\}\frac{I_\nu}{h\nu}\left[N_2-\frac{g_2}{g_1}N_1\right]$$

$$\left.\frac{dN_2}{dt}\right|_{radiative}=-A_{21}N_2-\sigma(\nu)\frac{I_\nu}{h\nu}\left[N_2-\frac{g_2}{g_1}N_1\right]$$

$\sigma(\nu)$: stimulated emission cross sction $\left(L^2\right)\rightarrow$ $\sigma(\nu)=A_{21}\frac{\lambda_0^2}{8\pi n^2}g(\nu)$

Summary of rate equation for the narrow band radiation in the cavity

Condition: FWHM$\left(\rho(\nu)\right) << FWHM\left(g(\nu)\right)$

$$\left.\frac{dN_2}{dt}\right|_{radiative} = -A_{21}N_2 - \sigma(\nu)\frac{I_\nu}{h\nu}\left[N_2 - \frac{g_2}{g_1}N_1\right]$$

with $\sigma(\nu)$ the stimulated emission cross sction defined as

$$\sigma(\nu) = A_{21}\frac{\lambda_0^2}{8\pi n^2}g(\nu)$$

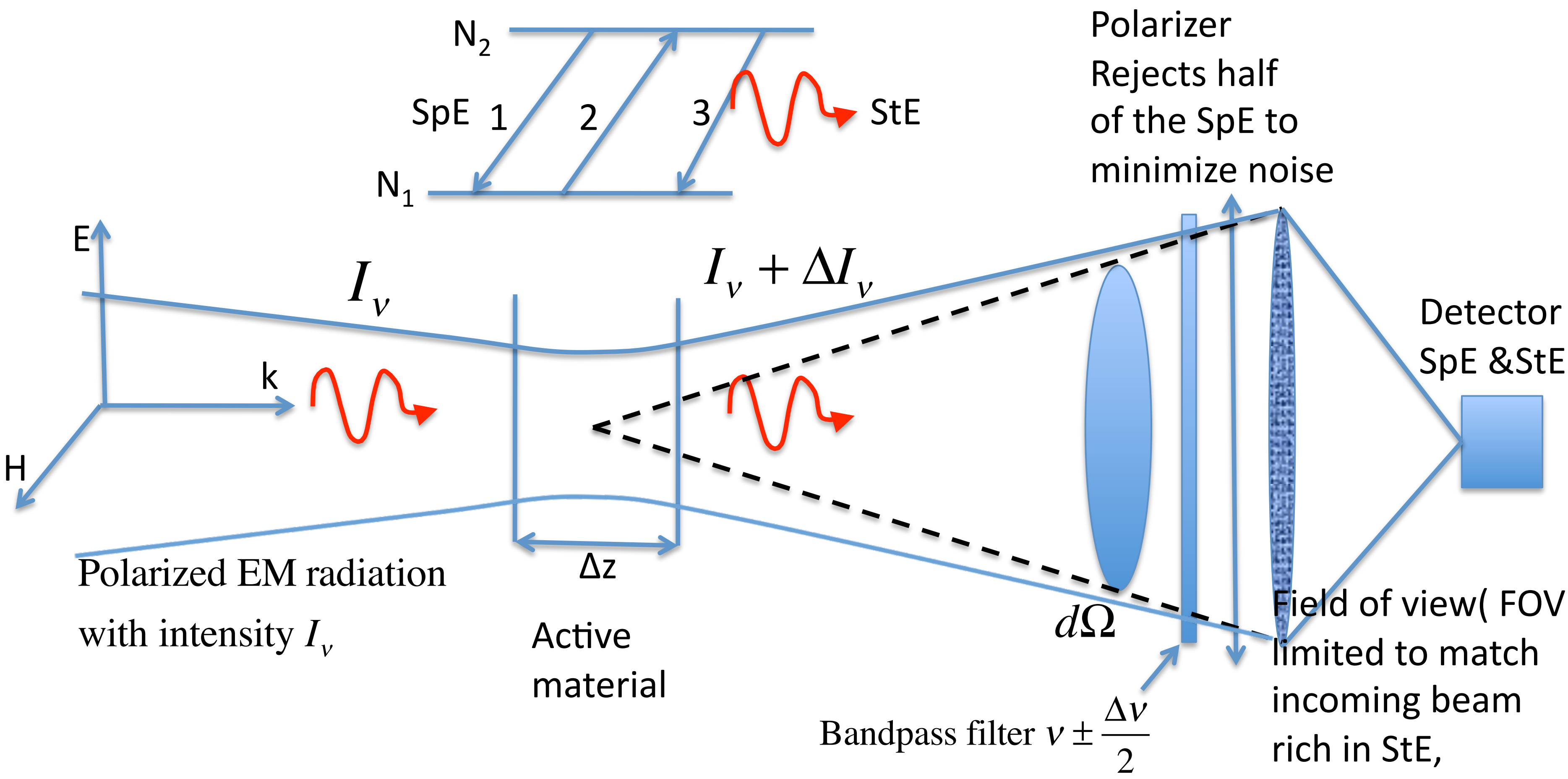
$\sigma(\nu)$ has dimension of the area or L^2

Conclusion: **the lineshape is the most important parameter in describing the interaction of the EM radiation with atoms.**

Amplification by an atomic system

- Goal: to describe the process of amplification (or attenuation) of EM energy by its interaction with atoms.
- Assumptions:
  - Population density of the atoms in various levels are not in thermal equilibrium
  - Populations are altered by external sources pumping (increases N_2) and various losses (decreases photon density in the system).
  - Spontaneous emission contributes to noise.

A Gedanken (thought) Experiment



Polarizer
Rejects half
of the SpE to
minimize noise

Detector
SpE &StE

Polarized EM radiation
with intensity I_ν

Active
material

Bandpass filter $\nu \pm \frac{\Delta\nu}{2}$

Field of view(FOV)
limited to match
incoming beam
rich in StE,
Only fraction of
SpE exists

Photon bookkeeping

$$
\begin{aligned}
\Delta I_\nu = \; & \overbrace{+h\nu}^{\substack{\text{1)pakage}\\ \text{of energy}}} \times \overbrace{B_{21}\frac{I_\nu}{(c/n)}}^{\substack{\text{2)power contributed by}\\ \text{each atom determined}\\ \text{by rate interaction}}} \times \overbrace{g(\nu)}^{\substack{\text{3)Effect of the}\\ \text{lineshape in}\\ \text{the process}}} \times \overbrace{1}^{\substack{\text{4)Effect of}\\ \text{polarization}\\ \text{of rdiation}}} \times \overbrace{1}^{\substack{\text{5)If the photon is}\\ \text{in the solid angle}\\ \text{determined}\\ \text{by the FOV}}} \times \overbrace{N_2\Delta z}^{\substack{\text{6)Number}\\ \text{of atoms in}\\ \text{the process}}} \\
& -h\nu \times B_{12}\frac{I_\nu}{(c/n)} \times g(\nu) \times 1 \times 1 \times N_1\Delta z \\
& +h\nu \times \mathrm{A}_{21}\Delta\nu \times g(\nu) \times \frac{1}{2} \times \frac{d\Omega}{4\pi} \times N_2\Delta z
\end{aligned}
$$

$$
\frac{\Delta I_\nu}{\Delta z} \rightarrow \frac{dI_\nu}{dz} = \underbrace{\left[\frac{h\nu}{c/n}\left(B_{21}N_2 - B_{12}N_1\right)g(\nu)\right]I_\nu}_{\text{A term due to absorption and stimulated emission}} + \underbrace{\frac{1}{2}\left[h\nu A_{21}N_2 g(\nu)\Delta\nu\frac{d\Omega}{4\pi}\right]}_{\substack{\text{Noise not dependent of the } I_\nu \\ \text{For now we neglect this term}}}
$$

Small signal gain coefficient

Using the Einstein conditions between the rate coefficients

$B_{12}g_1 = B_{21}g_2$ and $\mathrm{A}_{21} / B_{21} = 8\pi n^3 h\nu^3 / c^3$

we manupulate the change in intensity I_ν by thickness:

$$\frac{dI_\nu}{dz} = \left[A_{21}\frac{\lambda^2}{8\pi n^2}g(\nu)\right]\left[N_2 - \frac{g_2}{g_1}N_1\right]I_\nu \triangleq \underbrace{\gamma(\nu)}_{\textbf{gain coefficient}} \quad I_\nu$$

By definition the $\gamma(\nu)$ is called the **gain coefficient**.

Small signal gain $\gamma_0(\nu)$: is referred to the gain coefficient under the a condition that the I_ν does not change population density of the states 1 & 2.

All lasers start in small signal regime but soon I_ν starts affecting the N_2 and N_1. State populations become functions of I_ν.

Condition for amplification and gain

$$\frac{dI_\nu}{dz} = \underbrace{\left[A_{21}\frac{\lambda^2}{8\pi n^2}g(\nu)\right]}_{\substack{\text{Stimulated emission cross section}\\ \text{Always a positive term}}} \underbrace{\left[N_2 - \frac{g_2}{g_1}N_1\right]}_{\substack{\text{For increase of } I_\nu \text{ with}\\ \text{distance, we need this}\\ \text{term to be positive}}} I_\nu \triangleq \underbrace{\gamma(\nu)}_{\textbf{gain coefficient}} \quad I_\nu$$

$$\gamma(\nu) > 0 \text{ if } \left(N_2 - \frac{g_2}{g_1}N_1\right) > 0 \;\rightarrow \underbrace{N_2 > \frac{g_2}{g_1}N_1}_{\substack{\text{Positive gain or}\\ \text{amplification}}} \text{ and } \underbrace{\frac{dI_\nu}{dz} > 0}_{\text{Amplification}}$$

The gain condition is **contrary to Boltzman statistics**

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}e^{[-h\nu/k_BT]} \text{ for which } \underbrace{\frac{N_2}{g_2} < \frac{N_1}{g_1}}_{\text{Absorption}} \text{ and } \underbrace{\frac{dI_\nu}{dz} < 0}_{\text{absorption or extinction}}$$

Stimulated emission cross section

$$\frac{dI_\nu}{dz} = \underbrace{\left[A_{21}\frac{\lambda^2}{8\pi n^2}g(\nu)\right]}_{\text{Stimulated emission cross section}} \underbrace{\left[N_2 - \frac{g_2}{g_1}N_1\right]}_{\text{Effect of inversion density}} I_\nu \triangleq \underbrace{\gamma(\nu)}_{\textbf{gain coefficient}} I_\nu$$

$$\sigma(\nu) = \underbrace{A_{21}\frac{\lambda^2}{8\pi n^2}}_{\text{effect of the atom}} \underbrace{g(\nu)}_{\text{Effect of the environment}}$$

$$\left.\begin{aligned} &\sigma(\nu_0) = 10^{-16}\,cm^2 \\ &N_2 - \frac{g_2}{g_1}N_1 = 10^{14}\,cm^3 \end{aligned}\right\}\gamma(\nu) = 0.01cm^{-1} \text{ or a gain of 1\% per cm}$$

The absorption cross section is related to the stimulated emission cross section by: $\sigma_{abs}(\nu) = (g_2 / g_1)\sigma_{stim}(\nu)$

Solving for the intensity

We can find the intensity by solving this simple linear differential eq.

$$\frac{dI_\nu}{dz} = \gamma(\nu) I_\nu \rightarrow I_\nu(z) = I_\nu(0) e^{\gamma_0(\nu) z}$$

$$I_\nu(z) = G_0(\nu) I_\nu(0)$$

$G_0(\nu)$: Smal signal power gain of an amplifier length d

$$G_0(\nu) = e^{\gamma_0(\nu) d}$$

$$\gamma(\nu) = A_{21} \frac{\lambda^2}{8\pi n^2}\left(N_2 - \frac{g_2}{g_1} N_1 \right) g(\nu)$$

$\gamma(\nu)$ is frequency dependent and $G_0(\nu)$ more so due to the exponent. This is a **narrow band amplifier** and **we will put this in a feedback loop to gain oscillation.**

Line Broadening Mechanisms
Homogeneous vs inhomogeneous broadening

- Homogeneous line shape:
  - All molecules behave in the same way
  - Lorentz line shape
- Examples are:
  - Pressure broadening (collision)
  - Natural lifetime broadening
  - Transit time broadening
- Heissenberg's uncertainty principle

Line Broadening Mechanisms
Homogeneous vs. inhomogeneous broadening

- Inhomogeneous line shape:
- All molecules behave differently (distribution)
  - Gaussian line shape
- Examples are:
  - Doppler broadening
  - Power broadening

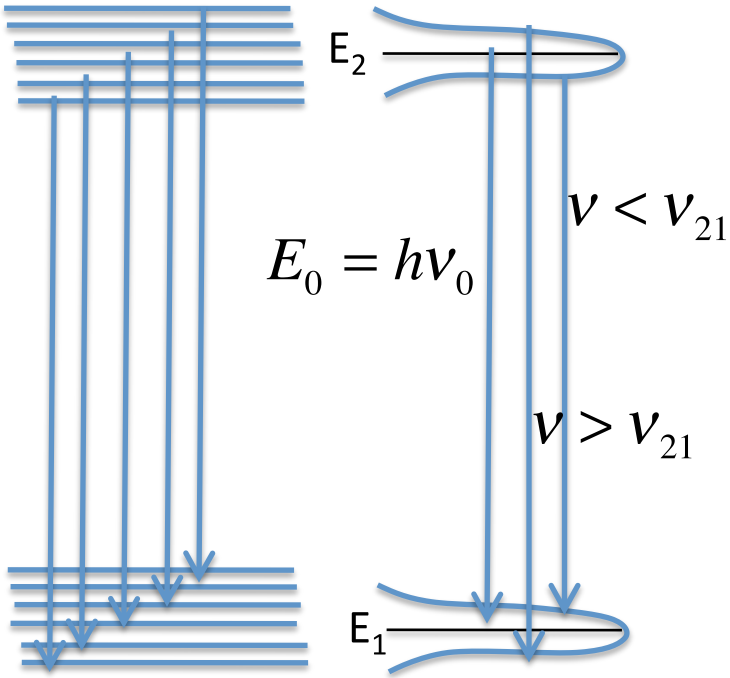
Lifetime broadening
A homogeneous mechanism

For simplicity we will only consider the **symmetric** case of the level broadening with **Lorentzian distribution**.

$$P_i(E) = \frac{\Delta E_i}{2\pi\left[(E - E_i)^2 + (\Delta E_i / 2)^2\right]}$$

$g(h\nu)$: the lineshape is the joint probability of all posssible transitions.

$$g(h\nu) = \int_{-\infty}^{\infty} P_2(E) P_1(E) dE$$



Lifetime broadening
A homogeneous mechanism

$P_i(E)$: Probability of a state being in E_i to $E_i + dE_i$ interval.

$N_1 P_1(E)dE$: # of atoms with energy E in the interval around E_1

$N_2 P_2(E)dE$: # of atoms with energy E in the interval around E_2

$$P_1(E) = \frac{\Delta E_1}{2\pi\left[(E-E_1)^2 + (\Delta E_1/2)^2\right]} \text{ with } \left\{\begin{array}{c} \Delta E_1 = FWHM \text{ of } P_1(E) \\ \Delta E_1 << (E_2 - E_1) \end{array}\right.$$

$$P_2(E) = \frac{\Delta E_2}{2\pi\left[(E-E_2)^2 + (\Delta E_2/2)^2\right]} \text{ with} \left\{\begin{array}{c} \Delta E_2 = FWHM \text{ of } P_2(E) \\ \Delta E_2 << (E_2 - E_1) \end{array}\right.$$

$\Delta E_{1,2} << (E_2 - E_1)$ Means the levels are well separated

Spontaneous emission of a photon hν will happen when there exists a combination of levels around E_1 and E_2 that can allow the radiation.

Lifetime broadening: Change of variables & finding lineshape g(hv)

$$g(h\nu)=\int_{-\infty}^{\infty}P_2(E)P_1(E)dE$$

For band 1: $E=x+E_1$ and $a=\Delta E_1/2$

For band 2: $E=x+E_1+h\nu$ and $\mathrm{b}=\Delta E_2/2$

And we define: $\delta=h\nu\text{ - }(E_2-E_1)$

$$g(h\nu)=\int_{-\infty}^{\infty}\left\{\frac{\Delta E_1}{2\pi}\frac{1}{\left[x^2+a^2\right]}\right\}\left\{\frac{\Delta E_2}{2\pi}\frac{1}{\left[(x+\delta)^2+b^2\right]}\right\}dx$$

Evaluate the integral (page 192 Verdeyene)

$$g(h\nu)=\frac{1}{2\pi}\frac{\Delta E_2+\Delta E_1}{\left[h\nu-(E_2-E_1)\right]^2+\left[(\Delta E_2+\Delta E_1)/2\right]^2}$$

Lifetime broadening: g(hv)

$$g\left(h\nu\right)=\frac{1}{2\pi}\frac{\Delta E_2+\Delta E_1}{\left[h\nu-\left(E_2-E_1\right)\right]^2+\left[\left(\Delta E_2+\Delta E_1\right)/2\right]^2}$$

Using $E=h\nu$ and relation of the energy levels to the lifetimes: $\Delta E.\Delta t=\hbar\ \left(\Delta E_i\bullet\tau_i=\hbar\right)$ and rearranging the $\mathrm{g}\left(\nu\right)$

$$g\left(\nu\right)=\frac{1}{2\pi}\frac{\Delta\nu}{\left[\left(\nu_0-\nu\right)^2+\left(\Delta\nu/2\right)^2\right]}\ \text{ with }\ \Delta\nu=\frac{1}{2\pi}\left\{\frac{1}{\tau_2}+\frac{1}{\tau_1}\right\}$$

Why we have limitted lifetime for the energy levels?

Energy transfer between the atoms or quenching collisions exists.

$$\frac{dN_2}{dt}=\underbrace{\left[\sum_{j<2}A_{2j}\right]N_2}_{rad}-\underbrace{k_2N_2}_{\text{quenching}}\triangleq-\frac{N_2}{\tau_2}\ \text{ where }\ \frac{1}{\tau_{rad}}=\sum_{j<2}A_{2j}$$

Lifetime broadening: g(hv)

$$\frac{dN_2}{dt}=\underbrace{\left[\sum_{j<2}A_{2j}\right]N_2}_{rad}-\underbrace{k_2N_2}_{\text{quenching}}\triangleq-\frac{N_2}{\tau_2}\ \text{ where }\ \frac{1}{\tau_{rad}}=\sum_{j<2}A_{2j}$$

$$\text{The broadening ratio=}\frac{\text{Ratio of decay from }2\rightarrow1}{\text{Sum of all decays from }2}\rightarrow\phi_{21}=\frac{A_{21}}{\sum A_{2j}+k_2}$$

The best we can do is to eleiminate the quenching so $k_2=0$

Due to ever-present nature of the various radiative processes,

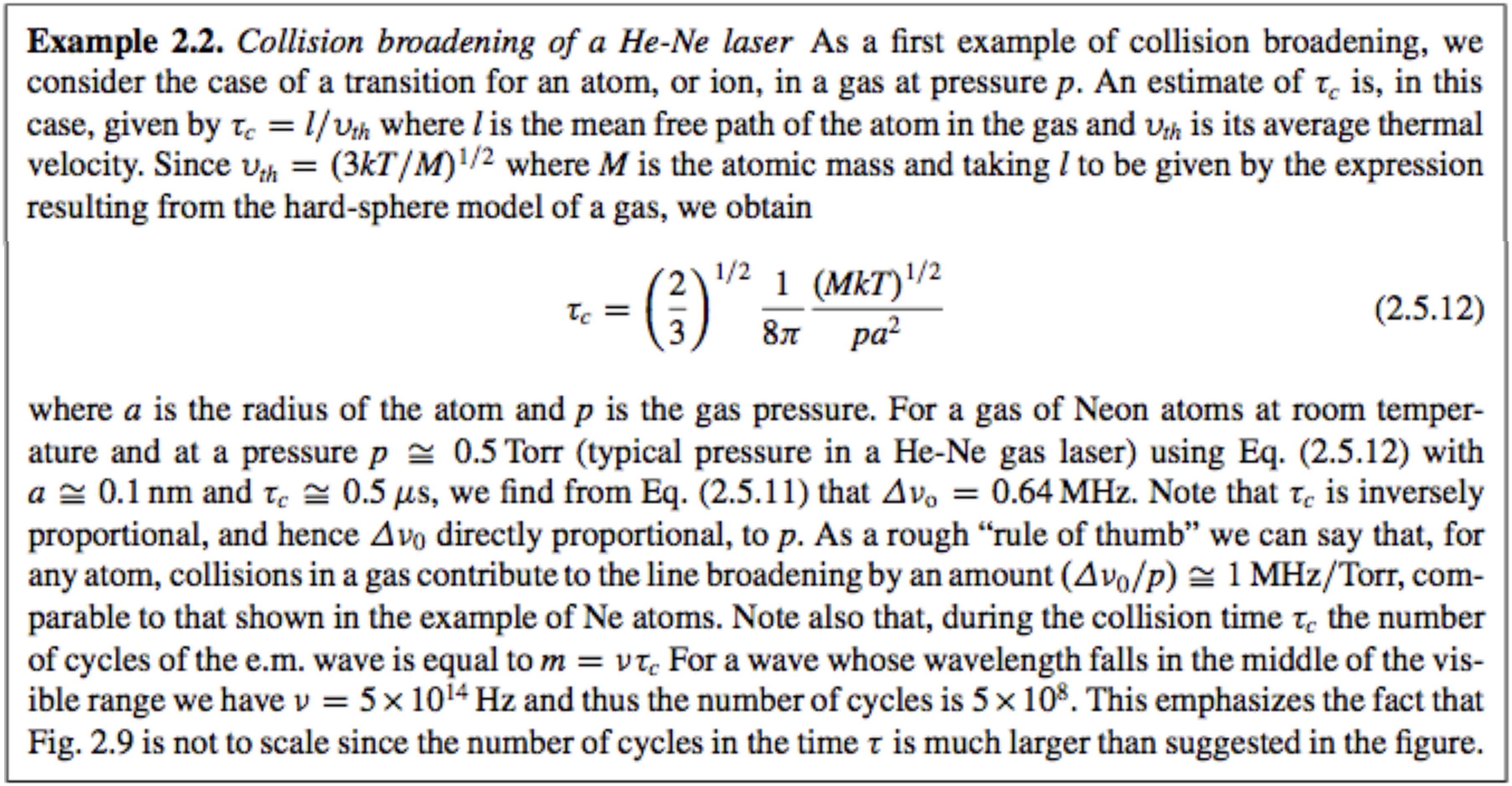
a non - zero linewidth allways exist that is so called "**natural linewidth**".

$$\Delta\nu_n=\frac{1}{2\pi}\{A_2+A_1\}=\ \frac{1}{2\pi}\left\{\frac{1}{\tau_2}+\frac{1}{\tau_1}\right\}$$

Other effects on line broadening are much more severe than the natural linewidth so we don't care much about it in practice.

Collision broadening

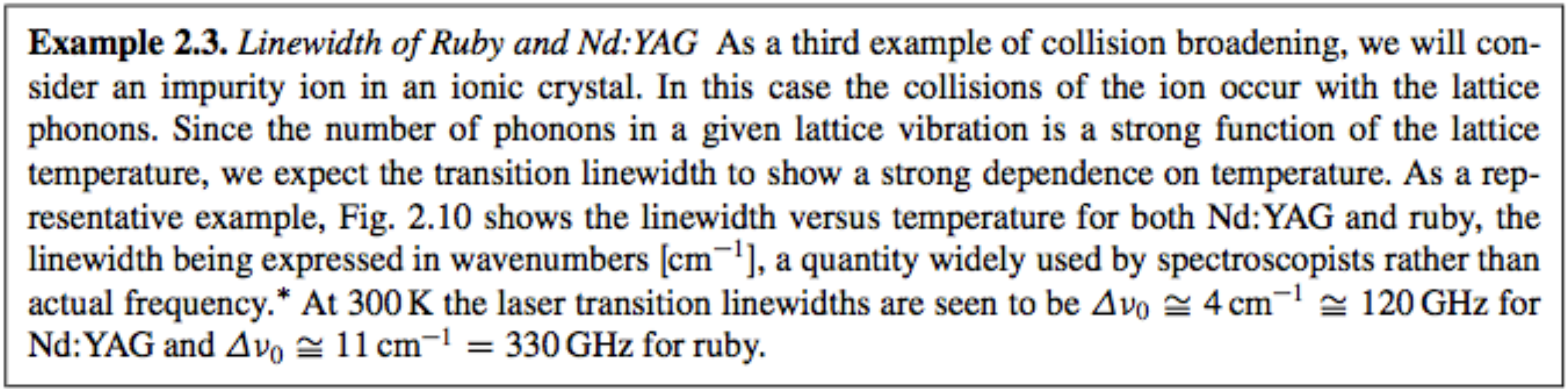
Example for collision broadening



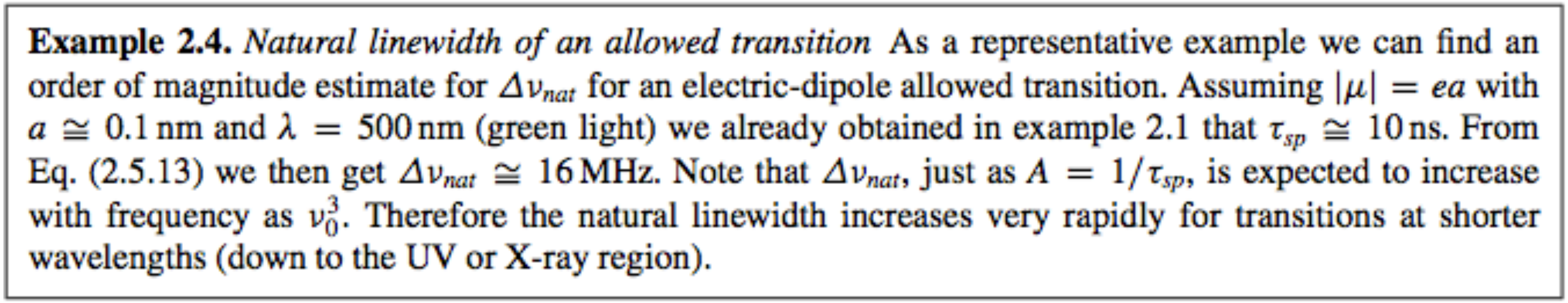
Example 2.2. *Collision broadening of a He-Ne laser* As a first example of collision broadening, we consider the case of a transition for an atom, or ion, in a gas at pressure p. An estimate of τ_c is, in this case, given by $\tau_c = l/\upsilon_{th}$ where l is the mean free path of the atom in the gas and υ_{th} is its average thermal velocity. Since $\upsilon_{th} = (3kT/M)^{1/2}$ where M is the atomic mass and taking l to be given by the expression resulting from the hard-sphere model of a gas, we obtain

$$\tau_c = \left(\frac{2}{3}\right)^{1/2} \frac{1}{8\pi} \frac{(MkT)^{1/2}}{pa^2} \tag{2.5.12}$$

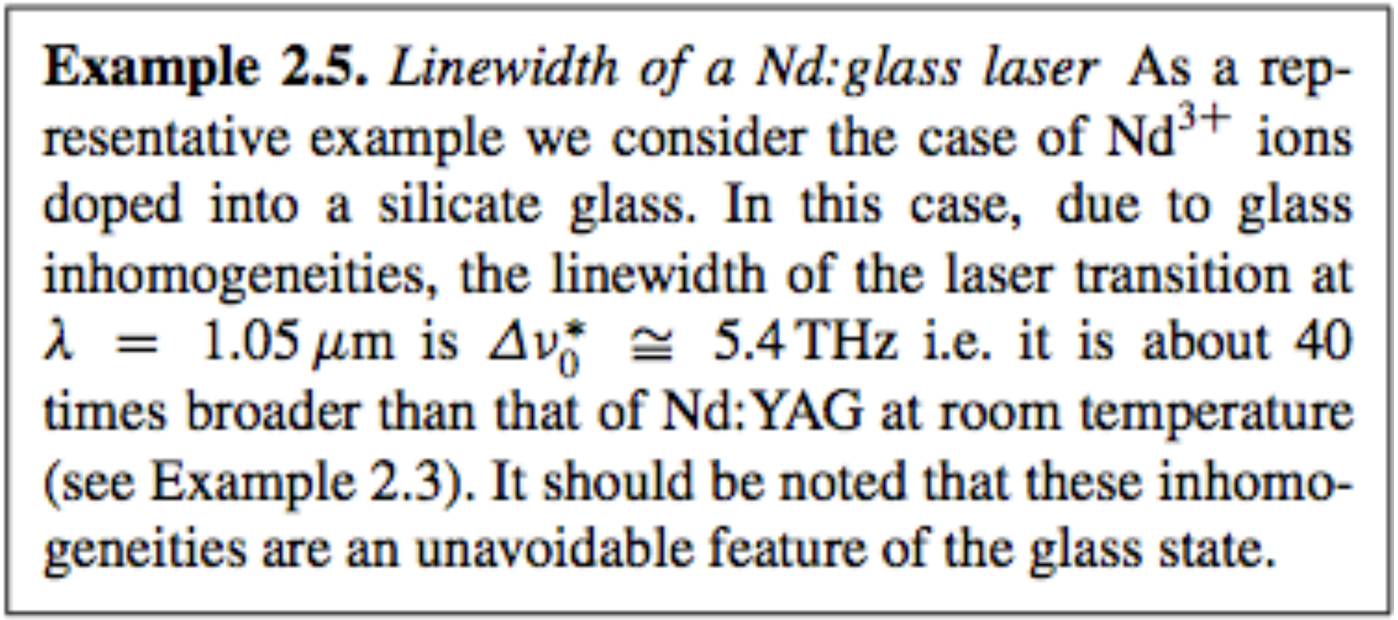
where a is the radius of the atom and p is the gas pressure. For a gas of Neon atoms at room temperature and at a pressure $p \cong 0.5$ Torr (typical pressure in a He-Ne gas laser) using Eq. (2.5.12) with $a \cong 0.1$ nm and $\tau_c \cong 0.5\ \mu$s, we find from Eq. (2.5.11) that $\Delta\nu_o = 0.64$ MHz. Note that τ_c is inversely proportional, and hence $\Delta\nu_0$ directly proportional, to p. As a rough "rule of thumb" we can say that, for any atom, collisions in a gas contribute to the line broadening by an amount $(\Delta\nu_0/p) \cong 1$ MHz/Torr, comparable to that shown in the example of Ne atoms. Note also that, during the collision time τ_c the number of cycles of the e.m. wave is equal to $m = \nu\tau_c$ For a wave whose wavelength falls in the middle of the visible range we have $\nu = 5 \times 10^{14}$ Hz and thus the number of cycles is 5×10^8. This emphasizes the fact that Fig. 2.9 is not to scale since the number of cycles in the time τ is much larger than suggested in the figure.



Example 2.3. *Linewidth of Ruby and Nd:YAG* As a third example of collision broadening, we will consider an impurity ion in an ionic crystal. In this case the collisions of the ion occur with the lattice phonons. Since the number of phonons in a given lattice vibration is a strong function of the lattice temperature, we expect the transition linewidth to show a strong dependence on temperature. As a representative example, Fig. 2.10 shows the linewidth versus temperature for both Nd:YAG and ruby, the linewidth being expressed in wavenumbers [cm^{-1}], a quantity widely used by spectroscopists rather than actual frequency.* At 300 K the laser transition linewidths are seen to be $\Delta\nu_0 \cong 4\,\mathrm{cm}^{-1} \cong 120\,\mathrm{GHz}$ for Nd:YAG and $\Delta\nu_0 \cong 11\,\mathrm{cm}^{-1} = 330\,\mathrm{GHz}$ for ruby.



Example 2.4. *Natural linewidth of an allowed transition* As a representative example we can find an order of magnitude estimate for $\Delta\nu_{nat}$ for an electric-dipole allowed transition. Assuming $|\mu| = ea$ with $a \cong 0.1\,\text{nm}$ and $\lambda = 500\,\text{nm}$ (green light) we already obtained in example 2.1 that $\tau_{sp} \cong 10\,\text{ns}$. From Eq. (2.5.13) we then get $\Delta\nu_{nat} \cong 16\,\text{MHz}$. Note that $\Delta\nu_{nat}$, just as $A = 1/\tau_{sp}$, is expected to increase with frequency as ν_0^3. Therefore the natural linewidth increases very rapidly for transitions at shorter wavelengths (down to the UV or X-ray region).



Example 2.5. *Linewidth of a Nd:glass laser* As a representative example we consider the case of Nd^{3+} ions doped into a silicate glass. In this case, due to glass inhomogeneities, the linewidth of the laser transition at $\lambda = 1.05\,\mu m$ is $\Delta\nu_0^* \cong 5.4\,THz$ i.e. it is about 40 times broader than that of Nd:YAG at room temperature (see Example 2.3). It should be noted that these inhomogeneities are an unavoidable feature of the glass state.

Inhomogeneous broadening

- Some of the shifts to the characteristic frequency of the atoms do not affect all of them in the same way such as
  - Isotope effect (same atomic number different atomic mass)
  - Doppler effect (same mass but different velocities)
  - Nuclear spin,
    - Hyperfine splitting separation of the lines due to coupling of angular momentum
    - Zeeman splitting of the lines due to external magnetic field
    - Stark shift, due to interaction of the levels with local lattice field
- Most of the time combination of these effects create asymmetric lineshapes which make the math more challenging.
- For example the 243.7 nm transition of the Hg atom is only 15 GHz due to combination of several effects.

Doppler broadening important in gas phase

due to variation in thermal velocities of the molecules

$\nu_0^{'} = \nu_0\left(1+\frac{V_z}{c}\right) \rightarrow \nu_0^{'} - \nu$ goes into the lineshpe

ν_0 : Emission frequency of atoms in their rest reference frame.

$\nu_0^{'}$: Emission frequency of atoms in the lab reference frame.

V_z : velocity of the atoms towards the observer

The homogeneous line width of those atoms with velocity V_z

$$g\left(V_z,\nu\right) = \frac{\Delta\nu_h}{2\pi\left[\left(\nu-\nu_0-\nu_0 V_z/c\right)^2+\left(\Delta\nu_h/2\right)^2\right]}$$

Fraction of atoms moving in the z direction with velocity

between V_z and $\mathrm{V}_z + dV_z$: $\quad \dfrac{dN}{N} = \left(\dfrac{M}{2nkT}\right)^{1/2} e^{-\frac{MV_z^2}{2k_BT}} dV_z$

$g\left(\nu\right) = \int_{-\infty}^{\infty} g\left(V_z,\nu\right)\dfrac{dN}{N}$ integration over all the velocities

Doppler broadening important in gas phase

$$g(\nu)=\left(\frac{M}{2nkT}\right)^{1/2}\int_{-\infty}^{\infty}\left\{\frac{\Delta\nu_h}{2\pi\left[\left(\nu-\nu_0-\nu_0V_z/c\right)^2+\left(\Delta\nu_h/2\right)^2\right]}\right\}e^{-\frac{MV_z^2}{2k_BT}}dV_z$$

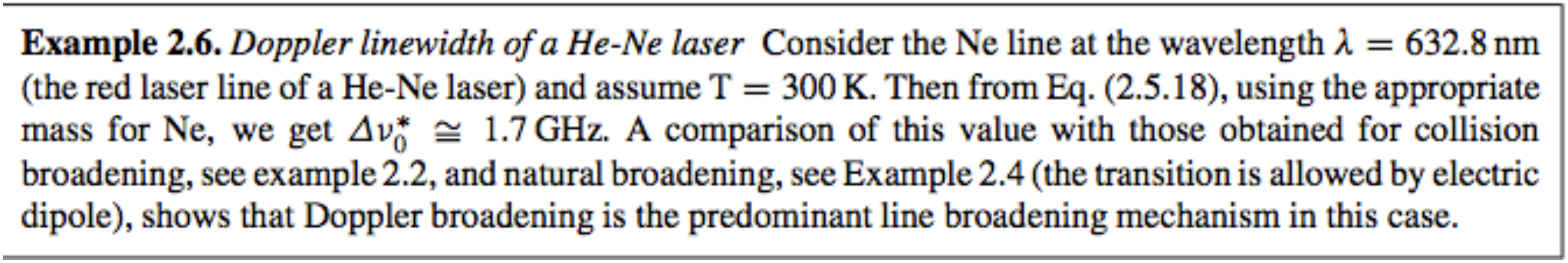
$$L(x-x')=\frac{\Delta x}{2\pi\left[\left(x-x'\right)^2+\left(\Delta x/2\right)^2\right]}\to\delta(x-x')$$

$$g(\nu)=\left(\frac{M}{2nkT}\right)^{1/2}\int_{-\infty}^{\infty}\delta\left(\nu-\nu_o-\frac{\nu_0V_z}{c}\right)e^{-\frac{MV_z^2}{2k_BT}}d\left(\frac{V_z\nu_0}{c}\right)\frac{c}{\nu_0}$$

$$g(\nu)=\frac{1}{\nu_0}\left(\frac{Mc^2}{2nkT}\right)^{1/2}e^{-\frac{Mc^2}{2k_BT}\left(\frac{\nu-\nu_0}{\nu_0}\right)^2}$$

$$\frac{Mc^2}{2k_BT}\left(\frac{\nu_{+,-}-\nu_0}{\nu_0}\right)^2=\ln 2$$

Example Doppler broadening



Example 2.6. *Doppler linewidth of a He-Ne laser* Consider the Ne line at the wavelength $\lambda = 632.8\,\text{nm}$ (the red laser line of a He-Ne laser) and assume $T = 300\,\text{K}$. Then from Eq. (2.5.18), using the appropriate mass for Ne, we get $\Delta\nu_0^* \cong 1.7\,\text{GHz}$. A comparison of this value with those obtained for collision broadening, see example 2.2, and natural broadening, see Example 2.4 (the transition is allowed by electric dipole), shows that Doppler broadening is the predominant line broadening mechanism in this case.

Comparison of the broadenings

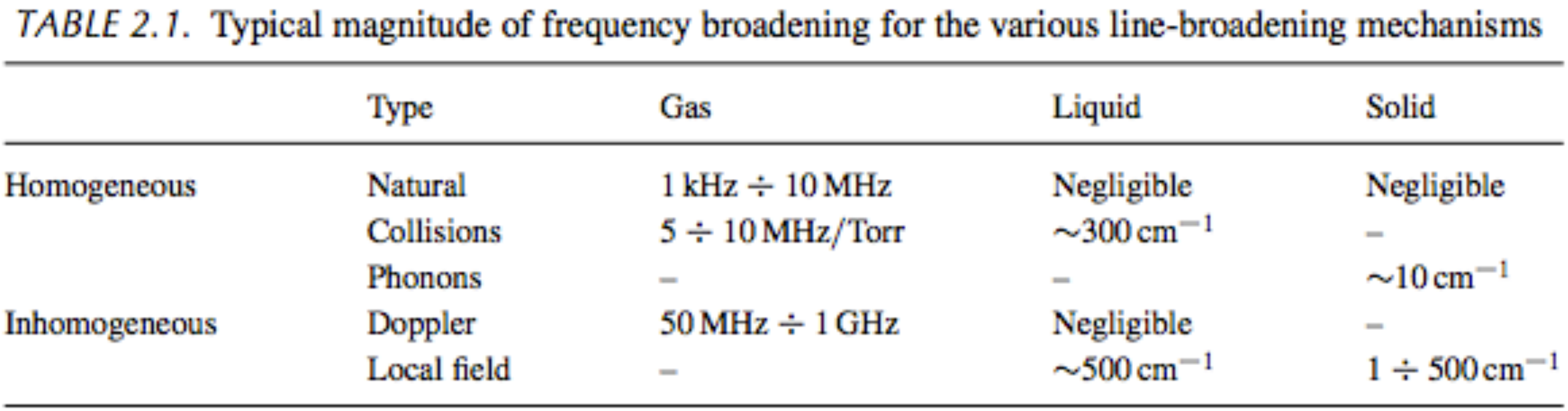


TABLE 2.1. Typical magnitude of frequency broadening for the various line-broadening mechanisms

	Type	Gas	Liquid	Solid
Homogeneous	Natural	1 kHz ÷ 10 MHz	Negligible	Negligible
	Collisions	5 ÷ 10 MHz/Torr	$\sim 300\,\mathrm{cm}^{-1}$	–
	Phonons	–	–	$\sim 10\,\mathrm{cm}^{-1}$
Inhomogeneous	Doppler	50 MHz ÷ 1 GHz	Negligible	–
	Local field	–	$\sim 500\,\mathrm{cm}^{-1}$	$1 \div 500\,\mathrm{cm}^{-1}$

(Svelto)

Cavity roundtrip