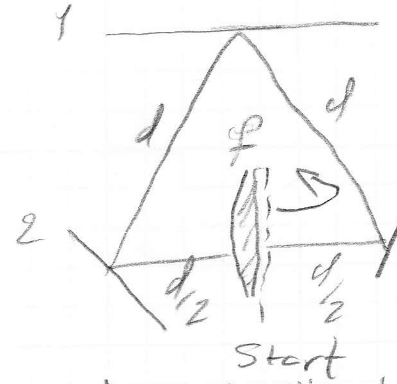
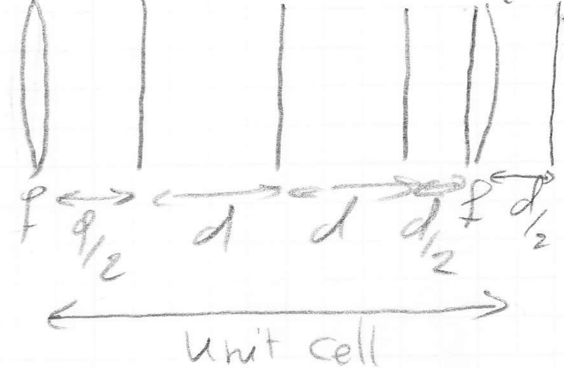
2.7 a) See the graph



$$T = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3d \\ 0 & 1 \end{bmatrix}$$

b) $$T = \begin{bmatrix} 1 & 3d \\ -\frac{1}{f} & -\frac{3d}{f}+1 \end{bmatrix}$$



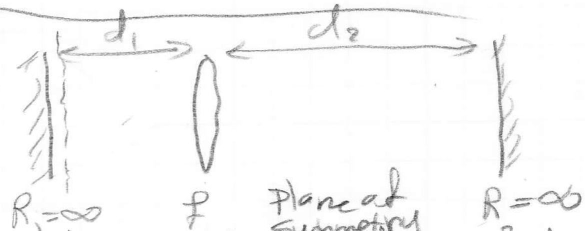
Test $AD - BC = 1$

$$1\left(1-\frac{3d}{f}\right) - \frac{3d}{f} = 1$$

c) For stability $-1 < \frac{A+D}{2} < 1$

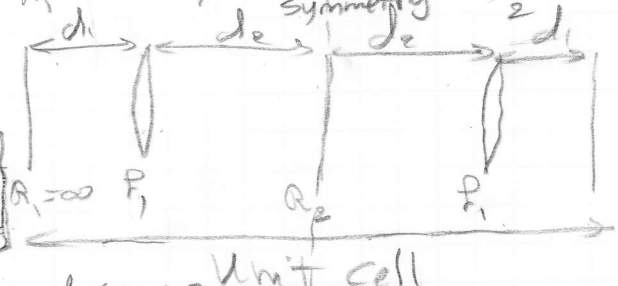
$$-1 \leq \frac{\cancel{1} - \frac{3d}{f} + \cancel{1}}{2} \leq 1 \Rightarrow 0 \leq \frac{d}{f} < \frac{4}{3}$$

2.8 a) See the diagram



b) c) $$T = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2d_2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix}$$



$$T = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ -\frac{1}{f} & 1-\frac{d_1}{f_1} \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{2d_2}{f} & d_1 + 2d_2(d_1 - \frac{1}{f}) \\ -\frac{1}{f} & 1-\frac{d_1}{f_1} \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{2d_2}{f} & d_1 + 2d_2(1-\frac{d_1}{f}) \\ -\frac{1}{f}(1-\frac{2d_2}{f}) - \frac{1}{f} & -\frac{1}{f}(d_1 + 2d_2(d_1-\frac{1}{f})) + (1-\frac{d_1}{f_1}) \end{bmatrix}$$

$$T = \begin{bmatrix} 1-\frac{2d_2}{f_1} - \frac{d_1}{f_1}(1-\frac{2d_2}{f_1}) - \frac{d_1}{f_1} & d_1 + 2d_2(1-\frac{d_1}{f}) - \frac{d_1}{f_1}(d_1 + 2d_2(d_1 - \frac{1}{f_1})) + d_1(1-\frac{d_1}{f_1}) \\ -\frac{1}{f_1}(1-\frac{2d_2}{f_1}) - \frac{1}{f_1} & -\frac{1}{f}(d_1 + 2d_2(1-\frac{d_1}{f_1})) + (1-\frac{d_1}{f_1}) \end{bmatrix}$$

2.8) d) $T = \begin{bmatrix} 1 - \frac{2d_1}{f} - \frac{2d_2}{f} + \frac{2d_1 d_2}{f^2} & \left(1-\frac{d_1}{f}\right)\left(2d_1 + 2d_2 - \frac{2d_1 d_2}{f}\right) \\ -\frac{2}{f}\left(1 - \frac{d_2}{f}\right) & 1 - \frac{2d_1}{f} - \frac{2d_2}{f} + \frac{2d_1 d_2}{f^2} \end{bmatrix}$

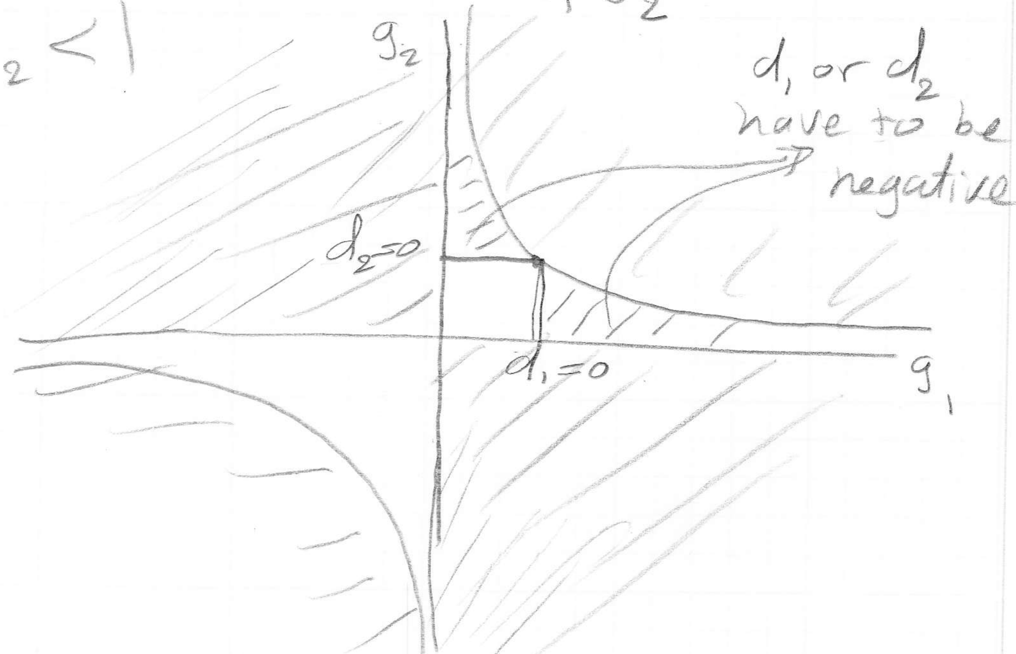
$A = D$ due to summetry plane

Stability condition: $\frac{A+D+2}{4} = 1 - \frac{d_1}{f_1} - \frac{d_2}{f_2} + \frac{d_1 d_2}{f^2}$

$= \left(1 - \frac{d_1}{f}\right)\left(1 - \frac{d_2}{f_2}\right)$

$= g_1 g_2$

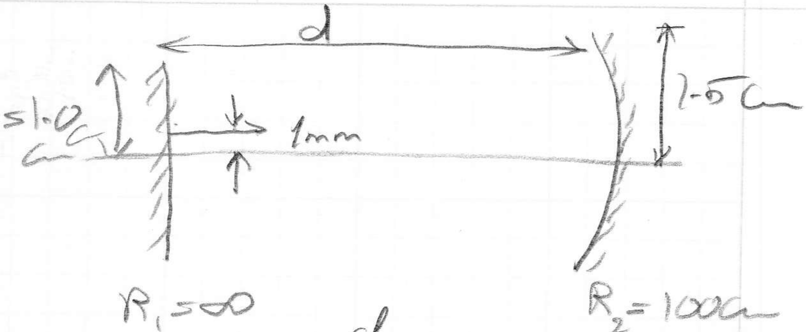
$0 \leqslant g_1 g_2 < 1$



2.10) Unstable cavity

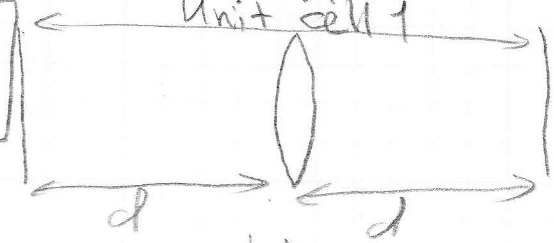
$r'(0) = 0$

$r(0) = \frac{R_2}{10}$



$\frac{d}{R} \leq 1.01$

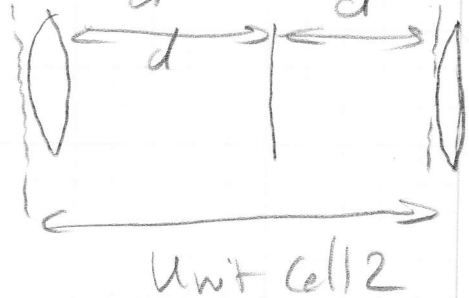
a) $T_1 = \begin{bmatrix} 1 - \frac{d}{f} & d + d(1 - \frac{d}{f}) \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{bmatrix}$



b) $r_s = r_a (F_+)^s + r_b (F_-)^s$

Values of F_+ & F_- (unit cell 1)

r_a & r_b



$$r_{s+2} - 2\,\frac{A+D}{2}\, r_{s+1} + r_s = 0$$

$$F_1 = \frac{A+D}{2} + \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{1/2} \qquad r_a = \frac{1}{F_1 + F_2}\left[a(F_1 - A) - Bm\right]$$

$$F_2 = \frac{A+D}{2} - \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{1/2} \qquad r_b = \frac{1}{F_2 - F_1}\left[a(F_2 - A) - Bm\right]$$

$$\frac{A+D}{2} = \frac{1}{2}\left(1 - \frac{d}{f} + 1 - \frac{d}{f}\right) = 1 - \frac{d}{f} = 1 - \frac{2d}{R} = 1 - 2(1.01)$$

$$\frac{A+D}{2} = 1 - 2.02 = -1.02 \qquad a = \frac{R_2}{10} = 10\text{cm}$$

$$B = A = 1 - \frac{d}{f} = 1 - 2.02 = -1.02 \qquad m = 0$$

$$F_1 = -1.02 + ((-1.02)^2 - 1)^{1/2} \Rightarrow \boxed{F_1 = -0.8190}$$

$$F_2 = -1.02 - ((-1.02)^2 - 1)^{1/2} \Rightarrow \boxed{F_2 = -1.2210}$$

2.10 Continued

$$r_a = \frac{1}{F_2 - F_1}\left[a(F_2 - A) - Bm\right] = 0.5\times10^{-2}$$

$$r_b = \frac{1}{F_1 - F_2}\left[a(F_1 - A) - Bm\right] = 0.5\times10^{-2}$$

c) # of passes before missing the flat mirror

$$r_s = r_a F_1^{\,s} + r_b F_2^{\,s} = (0.5\times10^{-2})\left[(0.819)^s + (-1.221)^s\right]$$

$r_s \geq 1 \Rightarrow \boxed{S = 15}$ after 15 roundtrips

for the unit cell 1

d) For Unit Cell 2 $r_a = -0.4525$; $r_b = 0.5525$

$$r_s = (-.4525)(0.819)^s + (0.5525)(-1.221)^s$$

$S = 6$ $r_s = 1.694\,\text{cm} > 1.5\,\text{cm}$ the spherical mirror is missed after 6 roundtrips.

e) $P_i = 1\,\text{mW}$ S is number of round trips.

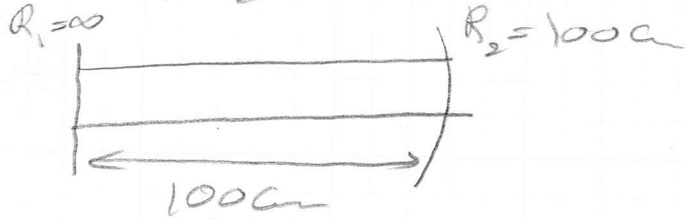
$G = 5$ per pass

$$P_{out} = P_i (G)^{2S+1} = (1\,\text{mW})(5)^{13}$$

each roundtrip raises the Gain two times once each way and on the final round before the scape there is one way since beam starts at flat mirror but scapes from the curved one

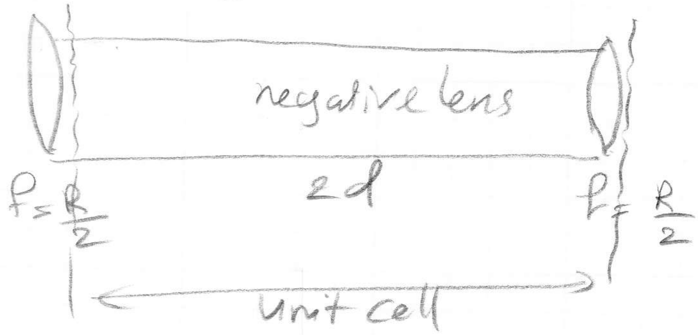
2.12 Gas discharge laser,
non-uniform gas (negative lens)
Boarderline stable cavity $0 < \frac{A+D}{2} \leq 1$

$\frac{d}{L} << 1$



To keep the pressure constant $d = R_2$
the density close to walls has to be larger than the center and that causes the $n_{center} < n_{edge}$ equivalent to a negative lens.

Finding the equivalent lens waveguide:



If
$$n(r) = n_0\left(1 + \frac{r^2}{2L^2}\right)$$
Then
$$T = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} \cosh\frac{2d}{L} & L\sinh\frac{2d}{L} \\ \frac{1}{L}\sinh\frac{2d}{L} & \cosh\frac{2d}{L} \end{bmatrix}$$
$$T = \begin{bmatrix} \cosh\frac{2d}{L} & L\sinh\frac{2d}{L} \\ -\frac{1}{f}\cosh\frac{2d}{L} + \frac{1}{L}\sinh\frac{2d}{L} & -\frac{L}{f}\sinh\frac{2d}{L} + \cosh\frac{2d}{L} \end{bmatrix}$$
Stability criteria
$$\frac{A+D+2}{4} = \frac{1}{4}\left(2 + 2\cosh\frac{2d}{L} - \frac{2L}{d}\sinh\frac{2d}{L}\right) = \frac{1}{2}\left(1 + \cosh\frac{2d}{L} - 2\frac{L}{2d}\sinh\frac{2d}{L}\right)$$
$\frac{2d}{L} = \theta$ if L is large θ is small we expand $\cosh\theta$ & $\sinh\theta$

2.12 Continued

$$\cosh\theta = 1 + \frac{\theta^2}{2} + \frac{\theta^4}{4!} \qquad \sinh\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$$

$$\frac{A+D+2}{4} = \frac{1}{2}\left\{1 + \cosh\theta - 2\frac{1}{\theta}\sinh\theta\right\}$$

$$= \frac{1}{2}\left\{1 + 1 + \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{2}{\theta}\left(\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right)\right\}$$

$$= \frac{1}{2}\left\{\cancel{2} + \frac{\theta^2}{2} + \cancel{\frac{\theta^4}{24}} - \cancel{2} - \frac{\theta^2}{3} + \cancel{\frac{\theta^4}{60}}\right\}$$

too small   too small

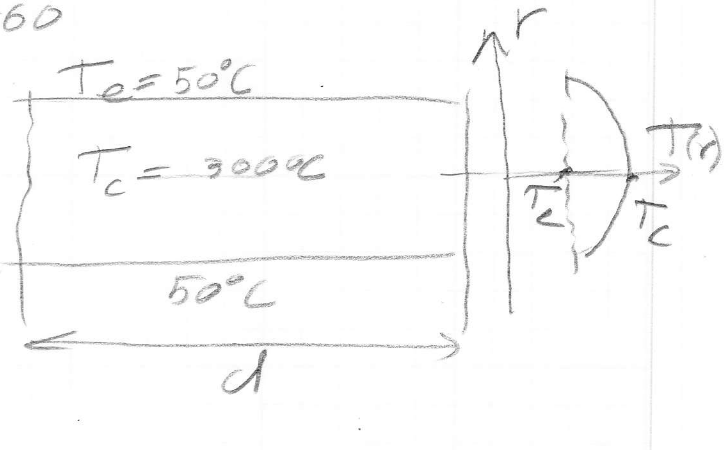
$$= \frac{1}{2}\left\{\frac{\theta^2}{6}\right\} = \frac{\theta^2}{12} + \text{a positive factor}$$

originally $\frac{A+D+2}{4} = 0$ now $\frac{A+D+2}{4} > 0$

$$\frac{A+D+2}{4} > \left(\frac{2d}{L}\right)^2 \frac{1}{12} = \frac{d^2}{3L^2}$$

Thus the cavity is more stable due to defocusing at the negative lens formed by gas heating.

2.13 $\begin{cases} P_0 = 1 \text{ torr} = \frac{1}{760} \text{ atm} = \frac{1.0 \times 10^5}{760} \text{ Pa} \\ T_0 = 23°C \end{cases}$



$P = N(r)kT(r) \neq f(r) = \text{cte}$

$\frac{d}{L} = ?$ at STP

$n(\text{helium}) = 1.000036$

$N_{total} = \text{Constant}$

$$n(r) = n_0\left(1 + \frac{1}{2}\left(\frac{r}{L}\right)^2\right) \leftarrow \text{Goal is to find } n(r)$$

$$T(r) = T_e + (T_c - T_e)\left(1 - \left(\frac{r}{a}\right)^2\right)$$

$$T(edge) = 50 = 50 + (300-50)\left(1 - \left(\frac{0.25}{a^2}\right)\right) \Rightarrow \frac{1}{250} = 1 - \frac{0.25}{a^2}$$

2.13 continued $\boxed{a = 5.094}$ $T(r) = 50 + 250\left(1 - \left(\frac{r}{5.094}\right)^2\right)$

Ideal gas law $PV = nRT$ for $P = \frac{N_A n}{V}\frac{R}{N_A}T = N_d kT \Rightarrow \boxed{P = N_d kT}$ # density

$N_{d,STP}$ number density at STP $P = 1\,atm = 1.0\times10^5\,Pa$

$T = 273K$

$$N_{d,STP} = \frac{1.0\times10^5\,Pa}{(1.38\times10^{-23}\,\frac{J}{K})(273\,K)} = 2.69\times10^{25}/m^3$$

$$\boxed{N_{d,STP} = 2.69\times10^{19}/cm^3}$$

number density at $P = 1$ Torr or $\frac{1}{760}$ atm and 23°C

$$N_0 = N_{d,STP}\frac{P_0}{P_{STP}}\frac{T_{STP}}{T_0} = 2.69\times10^{19}/cm^3\ \frac{1/760\ atm}{1\ atm}\ \frac{273\,K}{296\,K}$$

$\boxed{N_0 = 2.36\times10^{16}/cm^3}$ at 1 Torr & 23°C or initial gas condition

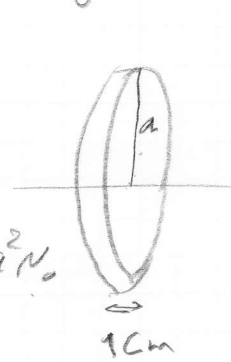
$$N(r)kT(r) = Const \Rightarrow N(r)kT(r) = N_c kT_c$$

$$N(r) = \frac{N_c T_c}{T(r)} = \frac{N_c}{\frac{T_w + (T_c - T_e)\left(1 - \left(\frac{r}{a}\right)^2\right)}{T_c}}$$

$$\boxed{N(r) = \frac{N_c}{1 - \left[\frac{T_c - T_e}{T_c}\right]\left(\frac{r}{a}\right)^2}}$$ number density function of $\underline{\underline{r}}$

# of atoms in a ring of 1cm long $= \pi a^2 N_0$

It has to be equal to $\int_0^a N(r)\,2\pi r\,dr = \pi a^2 N_0$



1cm

$$2\pi\int_0^a \frac{N_c\ r\,dr}{1 - \left(\frac{T_c - T_e}{T_c}\right)\left(\frac{r}{a}\right)^2} = \pi N_c\int_0^a \frac{a^2}{-\alpha}\frac{du}{u} = \pi N_c\frac{a^2}{-\alpha}\ln u\Big]_0^a$$

$u = 1 - \frac{\alpha}{a^2}r^2$ with $\alpha = \frac{T_c - T_w}{T_c}$

2.13 continued

$$\int_0^a N(r)\,2\pi r\,dr = \pi N_c \frac{a^2}{\frac{T_c - T_e}{T_c}} \ln\frac{T_c}{T_e} = \pi a^2 N_0$$

$$\boxed{N_c = \frac{T_c - T_e}{T_c}\frac{1}{\ln T_0/T_e} N_0} \Rightarrow$$ Number density at the center

$$N_c = \frac{300-50}{300}\frac{1}{\ln\frac{300}{50}} N_0 = 0.834 \times 2.36\times10^{16}/cm^3$$

$$\boxed{N_c = 2.7\times10^{16}/cm^3} \Rightarrow \boxed{N(r) = \frac{2.7\times10^{16}/cm^3}{1-\left(\frac{T_c-T_e}{T_c}\right)\left(\frac{r}{a}\right)^2}}$$

Now we can calculate the $n(r)$

$$n(r) - 1 = 2\pi\alpha N(r)$$

We only need the specific refractivity of the gas.

For He at STP: $2\pi\alpha = \frac{n_{STP}-1}{N_0} = \frac{1.000036-1}{2.69\times10^{19}/cm^3} = 1.338\times10^{-24}\,cm^3$

N_0 → # density of the gas

Now $n(r) = (2\pi\alpha)\,N(r) + 1$

$$n(r) = (2\pi\alpha)\,N_c\left(1 - \left(\frac{T_c-T_e}{T_c}\right)\left(\frac{r}{a}\right)^2\right)^{-1} + 1$$

$$= 2\pi\alpha N_c\left(1 + \frac{T_c-T_e}{T_c}\left(\frac{r}{a}\right)^2\right) + 1$$

$$n(r) \simeq 1.58\times10^{-8}\left(\frac{r}{a}\right)^2 + 1 \triangleq 1 + \frac{r^2}{2L^2}$$

$$\frac{1.58\times10^{-8}}{a^2} = \frac{1}{2L^2} \Rightarrow L^2 = \frac{a^2}{2(1.58\times10^{-8})} = \frac{(5.099)^2}{2(1.58\times10^{-8})} = 8.22\times10^8$$

$$\boxed{L = 2.81\times10^3\,cm}$$ or 28.1 m

2.13 continued

For CO_2

$n_{STP} = 1.000449$

$2\pi\alpha = 1.67 \times 10^{-23}\ cm^3$

$N_r = 2.7 \times 10^{+18} / cm^3$ at 100 Torr

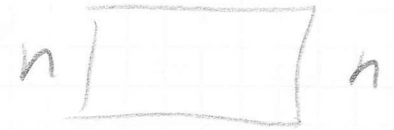
$n(r) = 1 + 1.96\left(\frac{r}{a}\right)^2 \triangleq 1 + \frac{r^2}{2L^2}$

$L = 79.7\ cm$ or $0.79\ m$

much shorter because the gas has higher pressuree more polarizability.

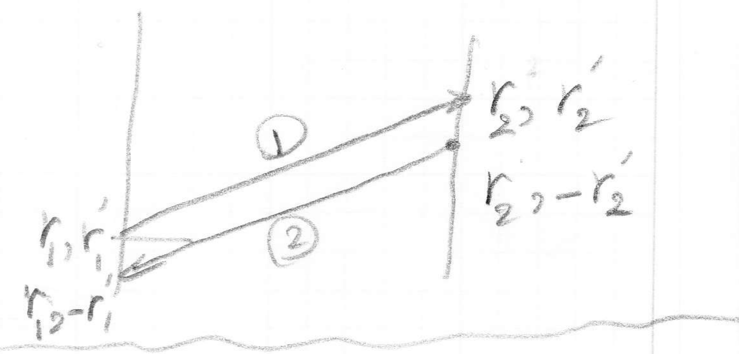
2.15 $AD-BC=1$ For a unit cell because one allways starts & ends at the same point so index does not change

$AD-BC=\frac{n}{n'}$



For a unit cell $n=n' \Rightarrow \boxed{AD-BC=1}$

2.22 $T_{12}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$



For the reverse direction the beam height stays the same but the angle changes.

(1) $\begin{bmatrix}r_2\\ r_2'\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}r_1\\ r_1'\end{bmatrix}$

$\underbrace{\begin{bmatrix}A' & B'\\ C' & D'\end{bmatrix}^{-1}}_{M^{-1}}\left(\begin{bmatrix}r_1\\ -r_1'\end{bmatrix}=\underbrace{\begin{bmatrix}A' & B'\\ C' & D'\end{bmatrix}}_{M}\begin{bmatrix}r_2\\ -r_2'\end{bmatrix}\right)$ See *

(2) $\begin{bmatrix}D' & -B'\\ -C' & A'\end{bmatrix}\begin{bmatrix}r_1\\ -r_1'\end{bmatrix}=\begin{bmatrix}r_2\\ -r_2'\end{bmatrix}$

1 & 2 $\begin{cases} r_2 = Ar_1 + Br_1' = D'r_1 + B'r_1' \\ r_2' = Cr_1 + Dr_1' = C'r_1 + A'r_1' \end{cases}$

$\Rightarrow D'=A,\ B'=B$
$C'=C,\ D=A' \Rightarrow \boxed{M=\begin{bmatrix}D & B\\ C & A\end{bmatrix}=T_{21}}$

*
$\underbrace{\begin{bmatrix}A' & B'\\ C' & D'\end{bmatrix}}_{M}^{-1}=\frac{1}{\det M}\begin{bmatrix}D' & -C'\\ -B' & A'\end{bmatrix}^{T}$

$M^{-1}=\begin{bmatrix}D' & -B'\\ -C' & A'\end{bmatrix}$

$MM^{-1}=\begin{bmatrix}D' & -B'\\ -C' & A'\end{bmatrix}\begin{bmatrix}A' & B'\\ C' & D'\end{bmatrix}$

$MM^{-1}=\begin{bmatrix}D'A'-B'C' & D'B'-B'D'\\ -C'A'+A'C' & D'A'-C'B'\end{bmatrix}$

$MM^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

HW 4 P168

2.23 $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ $T' = \begin{bmatrix} D & B \\ C & A \end{bmatrix}$

T T'

a) For roundtrip RTM we have: $T_{\text{roundtrip}} = T'T$

a) $$T_{RT} = \begin{bmatrix} D & B \\ C & A \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} DA+BC & 2DB \\ 2AC & CB+AD \end{bmatrix}$$

b) $$0 < \frac{(DA+BC)+(CB+AD)+2}{4} < 1$$

$$\frac{2DA+2BC+2}{4} = \frac{DA+BC+1}{2}$$
$$AD - BC = 1 \Rightarrow BC = AD - 1$$
$$\Rightarrow \frac{DA+AD\cancel{-1}\cancel{+1}}{2} = AD$$

$\boxed{0 < AD < 1}$ Stability condition

2.24 Unit cell $T = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ $T' = \begin{bmatrix} D_1 & B_1 \\ C_1 & A_1 \end{bmatrix}$ $T = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ $T' = \begin{bmatrix} D_1 & B_1 \\ C_1 & A_1 \end{bmatrix}$

Unit cell

$$T_{\text{unit cell}} = \underbrace{T'T}\,\underbrace{T'T}$$

$$T = \begin{bmatrix} DA+BC & 2BD \\ 2AC & CB+AD \end{bmatrix}\begin{bmatrix} DA+BC & 2BD \\ 2AC & CB+AD \end{bmatrix} = \begin{bmatrix} \overbrace{DA+BC}^{A_a} & \overbrace{2BD}^{B_a} \\ \underbrace{2AC}_{C_a} & \underbrace{CB+AD}_{D_a} \end{bmatrix}^2$$

$$M^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix}$$

a) $$\boxed{T = \begin{bmatrix} A_a^2 + B_aC_a & A_aB_a + B_aD_a \\ C_aA_a + D_aC_a & C_aB_a + D_a^2 \end{bmatrix}}$$

b) $$0 \leq \frac{A+D+2}{4} \leq 1 \Rightarrow 0 < \frac{A_a^2 + B_aC_a + C_aB_a + D_a^2 + 2}{4} < 1$$

$$B_aC_a = A_aD_a - 1 \Rightarrow \quad 0 < \frac{(A_a+D_a)^2}{4} < 1 \quad \text{with } A_a = D_a$$

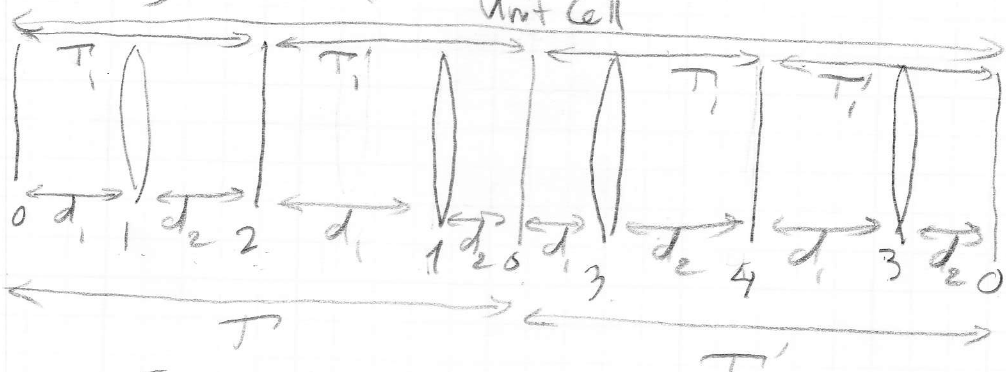
$$(A_a+D_a)^2 = (2A_a)^2 \quad \Rightarrow 0 < A_a^2 < 1 \Rightarrow -1 < A_a < 1$$

$$-1 < DA + BC < 1 \Rightarrow -1 < 2D_1A_1 - 1 < 1 \quad \text{or}$$

$$0 < 2D_1A_1 < 2 \Rightarrow \boxed{0 < D_1A_1 < 1}$$

2.24 (continued)

Unfolding the cavity



$$T_1 = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ -\frac{1}{f} & 1-\frac{d_1}{f} \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1-\frac{d_2}{f} & d_1 + d_2\left(1-\frac{d_1}{f}\right) \\ -\frac{1}{f} & 1-\frac{d_1}{f} \end{bmatrix}$$

Stability condition $0 < A_1 D_1 < 1$

$$0 < \left(1-\frac{d_2}{f}\right)\left(1-\frac{d_1}{f}\right) < 1 \Rightarrow \boxed{0 < g_1 g_2 < 1}$$

So it helps to think about the choice of Unit cell and how to divide the round trip to symmetric sections then use the method dueloped in problems 22, 23, 24