Chapter 2
Matrix Methods in Paraxial Optics

Lecture Notes for PHYS 168 Lasers based on
Pedrotti[3] & Verdeyen
Instructor: Nayer Eradat
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Matrix methods in paraxial optics

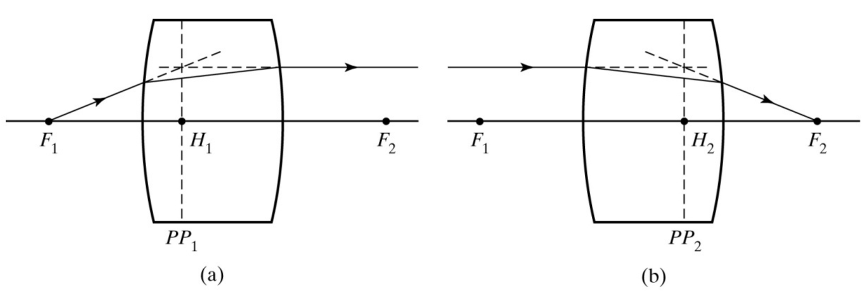
- Describing a single thick lens in terms of its cardinal points.
- Describing a single optical element with a 2x2 matrix.
- Analysis of train of optical elements by multiplication of 2x2 matrices describing each element.
- Computer ray-tracing methods, a more systematic approach

Cardinal points and cardinal planes

- Imaging properties of a thick lens can be deduced from the six cardinal points on its axis. Planes normal to the axis at the cardinal points are called cardinal planes. They are:
- First and second set of focal points and focal planes.
- First and second set of principal points and principal planes.
- First and second set of nodal points and nodal planes.

Focal points and focal planes

- Object focal plane: loci of the object points when image is at infinity
- Image focal plane: loci of the image points when object is at infinity
- Image and object focal points: intersection of the object and image focal planes with the optical axis.



Principal points and principal planes

- The rays determining the focal points change direction at their intersection with the principal planes.
- Principal points are at the intersection of the principal planes the optical axis.
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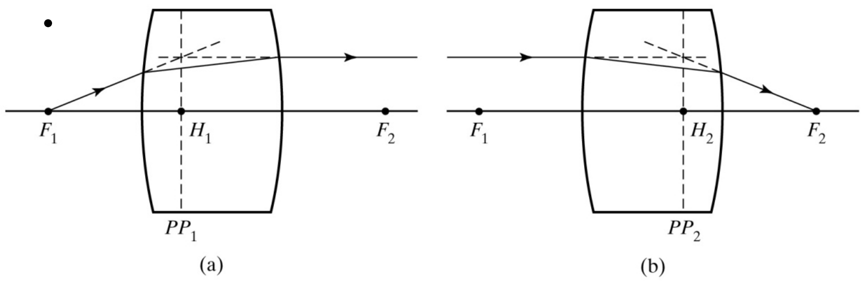
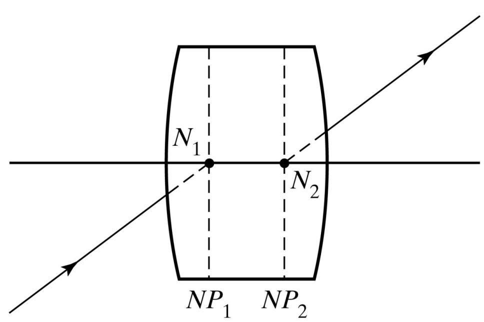
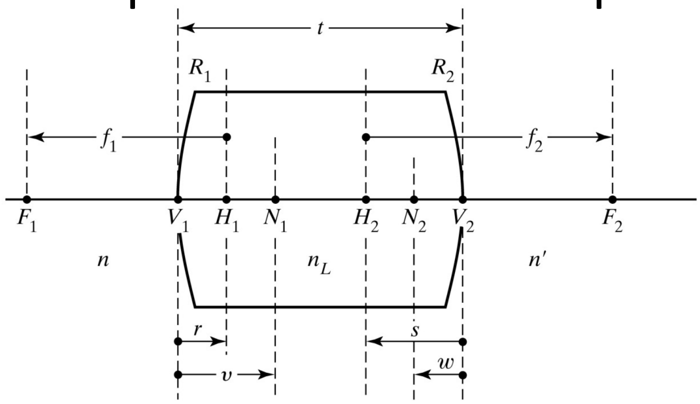


Nodal points and nodal planes

- Nodal points of a thick lens or any optical system permit correction to the ray that aims the center of the lens.
- Any ray that aims the first nodal point emerges from the second nodal point undeviated but slightly displaced.



Cardinal points and cardinal planes



- All the distances that are directed **to the left are negative (-)**
- All the distances that are directed **to the right are positive (+)**
- Notice that focal distances are not measured from the vertices

Basic equations for the thick lens

$$\frac{1}{f_1} = \frac{n_L - n'}{nR_2} - \frac{n_L - n}{nR_1} - \frac{(n_L - n)(n_L - n')}{nn_L}\frac{t}{R_1 R_2} \quad \text{and} \quad f_2 = -\frac{n'}{n} f_1$$

If $n = n'$ then $f_2 = -f_1$

Location of the principal planes:

$$r = \frac{n_L - n'}{n_L R_2} f_1 t;$$

$$s = \frac{n_L - n}{n_L R_1} f_2 t$$

The positions of the nodal points:

$$v = \left(1 - \frac{n'}{n} + \frac{n_L - n'}{n_L R_2} t\right) f_1; \quad w = \left(1 - \frac{n}{n'} + \frac{n_L - n}{n_L R_1} t\right) f_2$$

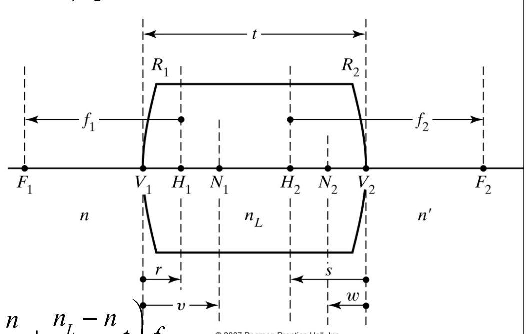


Image and object distances and lateral magnification:

$$-\frac{f_1}{s_o} + \frac{f_2}{s_i} = 1 \quad \text{and } m = -\frac{ns_i}{n's_o}$$

Basic equations for the thick lens

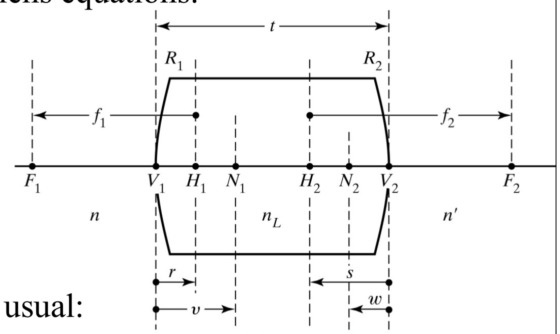
For an ordinary thin lens in air: $n = n' = 1$ and $r = v,\ s = w$

we arive at the usual thin lens equations:

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

$$m = -\frac{s_i}{s_o}$$

$$f = f_2 = -f_1$$

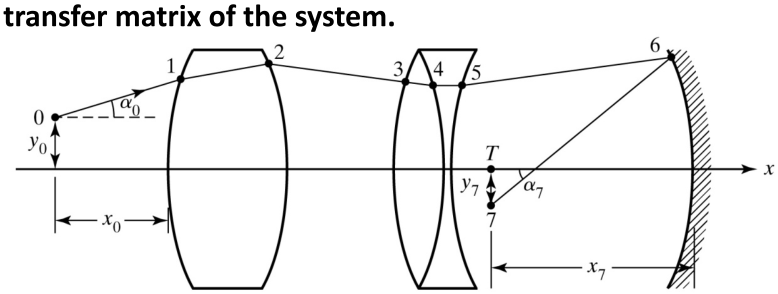


The sign convention is as usual:

real + and virtual – as long as the distances are measured relative to their corresponding principal planes.

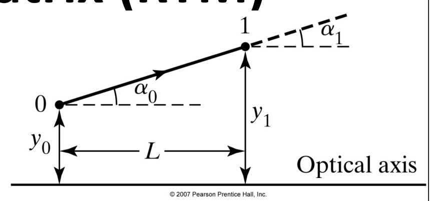
The matrix methods in paraxial optics

- For optical systems with many elements we use a systematic (programmable) approach called matrix method.
- For each ray as it progresses through the optical system we follow two parameters .
- **A ray is defined by its height y and its angle α with the optical axis**.
- **We can express y_{final} and α_{final} in terms of y_1 and α_1 multiplied by the transfer matrix of the system.**



The Ray Transfer Matrix (RTM)

$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\substack{\text{Ray Transfer Matrix} \\ (RTM)}} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix}$$



$r \triangleq y$ the **beam height** from the optical axis.

$r' \triangleq \alpha$ the **beam angle** with the optical axis.

One can show based on a thermodynamics arguments that

$$\det(RTM) = AD - BC = \frac{n_{in}}{n_{out}}$$

The general Ray-transfer matrix

Generalizing the matrix relationship for any number of translating, reflecting, refracting surfaces:

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = M \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} \quad \text{where } M = M_N M_{N-1} \cdots M_2 M_1$$

M is the **RTM of the optical system**.

Some rules :

Matrix multiplication is non-comutative: $\mathrm{M}_1 M_2 \neq M_2 M_1$

Matrix multiplication is ssociative: $(M_3 M_2) M_1 = M_3 (M_2 M_1)$

Order of operations from first the ray sees to the las one is **right to left**.

RTM of translation

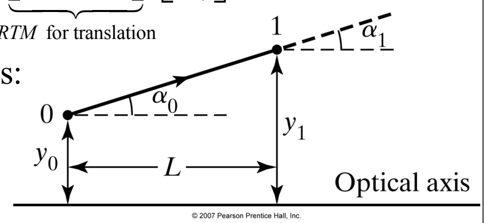
Simple tanslation of a ray in a **homogeneous** medium.

Translation from point 0 to 1 with **paraxial approximation***:

$\alpha_1 = \alpha_0$ and $y_1 = y_0 + L\tan\alpha_0 = y_0 + L\alpha_0$

We rewrite the equations:

$$\left.\begin{aligned} y_1 &= (1)y_0 + (L)\alpha_0 \\ \alpha_1 &= (0)y_0 + (1)\alpha_0 \end{aligned}\right\} \rightarrow \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}}_{RTM \text{ for translation}} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

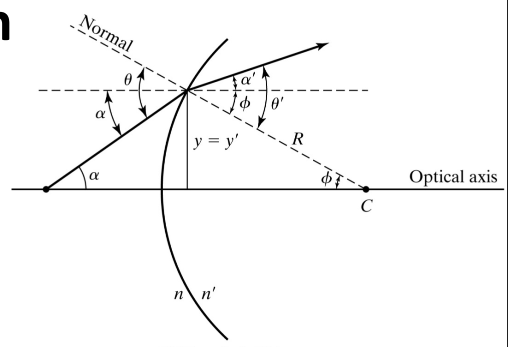


Combination of two translations:

$$\begin{bmatrix} 1 & L_1 + L_2 \\ 0 & 1 \end{bmatrix}$$

*Paraxial approxiamation: $\tan\alpha \simeq \sin\alpha \simeq \alpha(\text{rad})$

RTM of Refraction



Refraction of a ray at a spherical interface:

Ray coordinates before refraction (y, α)

Ray coordinates after refraction (y', α')

$\alpha' = \theta' - \phi = \theta' - \frac{y}{R}$ and $\alpha = \theta - \phi = \theta - \frac{y}{R}$

Paraxial form of Snell's law: $n\theta = n'\theta'$

$$\alpha' = \left(\frac{n}{n'}\right)\theta - \frac{y}{R} = \left(\frac{n}{n'}\right)\left(\alpha + \frac{y}{R}\right) - \frac{y}{R}$$

The approximate linear equations become:

$$\left.\begin{aligned} y' &= (1)y + (0)\alpha \\ \alpha' &= \left[\left(\frac{1}{R}\right)\left(\frac{n}{n'} - 1\right)\right]y + \left(\frac{n}{n'}\right)\alpha \end{aligned}\right\} \rightarrow \begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n}{n'} - 1\right) & \frac{n}{n'} \end{bmatrix}}_{\text{Ray-transfer matrix for refraction}} \begin{bmatrix} y \\ \alpha \end{bmatrix}$$

What happens if $R \rightarrow \infty$? $\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & n/n' \end{bmatrix}}_{\text{Refraction at a planar interface}} \begin{bmatrix} y \\ \alpha \end{bmatrix}$

RTM of reflection

Refraction of a ray at a spherical interface:

Ray coordinates before refraction $\left(\mathrm{y},\alpha\right)$

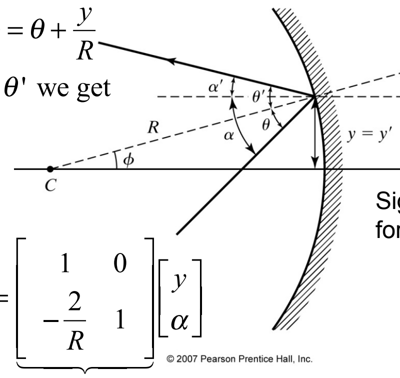
Ray coordinates after refraction $\left(\mathrm{y}',\alpha'\right)$

Goal: connect $\left(\mathrm{y}',\alpha'\right)$ to $\left(\mathrm{y},\alpha\right)$ by a ray transfer matrix

$\alpha' = \theta' - \phi = \theta' - \frac{y}{R}$ and $\alpha = \theta + \phi = \theta + \frac{y}{R}$

To eliminate θ and θ' we use $\theta = \theta'$ we get

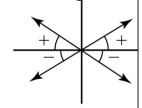
$$\alpha' = \theta - \frac{y}{R} = \alpha - \frac{2y}{R}$$



The desired equations become:

$$\left.\begin{aligned} \mathrm{y}' &= (1)\,y + (0)\,\alpha \\ \alpha' &= \left(-\frac{2}{R}\right) y + (1)\,\alpha \end{aligned}\right\} \rightarrow \begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}}_{\text{RTM for reflection}} \begin{bmatrix} y \\ \alpha \end{bmatrix}$$

Sign convention for the angles



The thick lens and thin lens matrices I

Goal: transfer matrix of a thick lens with different materials on sides.

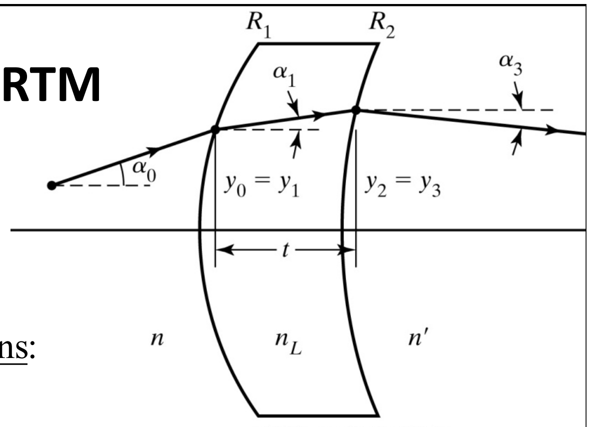
The operation consists of **two refractions** and **one translation**.

The radii of curvature are $(+)$ in this example.

$$\left.\begin{aligned} \begin{bmatrix} \mathrm{y}_1 \\ \alpha_1 \end{bmatrix} &= M_1 \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} \text{ for the first reflection} \\ \begin{bmatrix} \mathrm{y}_2 \\ \alpha_2 \end{bmatrix} &= M_2 \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} \text{ for the translation} \\ \begin{bmatrix} \mathrm{y}_3 \\ \alpha_3 \end{bmatrix} &= M_3 \begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} \text{ for the second reflectiont} \end{aligned}\right\} \rightarrow \begin{bmatrix} \mathrm{y}_3 \\ \alpha_3 \end{bmatrix} = \underbrace{M_3 M_2 M_1}_{\substack{M:\ \text{Transfer matrix} \\ \text{of the entire lens}}} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

Matrices operate on the ray in the same order in which the optical actins influense the ray.

The thick lens RTM



The RTM for a thick lens:

$M = M_3 M_2 M = R_2 T R_1$

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{n_L - n'}{n' R_2} & \dfrac{n_L}{n'} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{n - n_L}{n_L R_1} & \dfrac{n}{n_L} \end{bmatrix}$$

R represents a refraction and T represents a tsranslation matrix

The thin lens RTM

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{n_L - n'}{n' R_2} & \dfrac{n_L}{n'} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{n - n_L}{n_L R_1} & \dfrac{n}{n_L} \end{bmatrix}$$

For a thin lens $(t \to 0)$ in one environment $(n = n')$ the M becomes:

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{n_L - n}{n R_2} & \dfrac{n_L}{n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{n - n_L}{n_L R_1} & \dfrac{n}{n_L} \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{n_L - n}{n}\left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right) & 1 \end{bmatrix}$$

$$M = \underbrace{\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}}_{\text{RTM for a thin lens}} \quad \text{with} \quad \underbrace{\frac{1}{f} = \frac{n_L - n}{n}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)}_{\text{lensmaker's formula}}$$

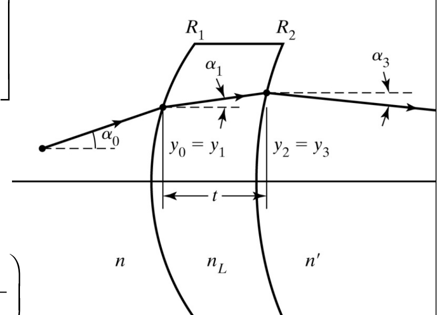
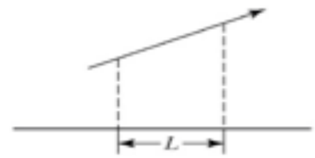


TABLE 18-1 SUMMARY OF SOME SIMPLE RAY-TRANSFER MATRICES

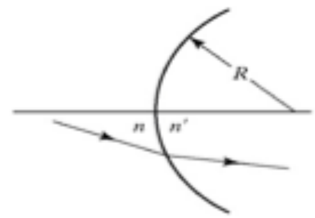
Translation matrix:

$$M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \mathfrak{T}$$



Refraction matrix, spherical interface:

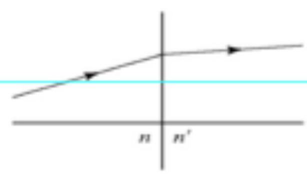
$$M = \begin{bmatrix} 1 & \not{L}\ 0 \\ \frac{n - n'}{Rn'} & \frac{n}{n'} \end{bmatrix} = \mathfrak{R}$$



$(+R)$: convex
$(-R)$: concave

Refraction matrix, plane interface:

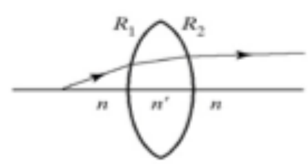
$$M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{bmatrix}$$



Thin-lens matrix:

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

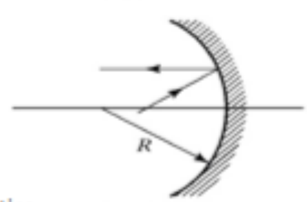
$$\frac{1}{f} = \frac{n' - n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$



$(+f)$: convex
$(-f)$: concave

Spherical mirror matrix:

$$M = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$



Matrix Methods in Paraxial Optics

$(+R)$: convex
$(-R)$: concave

Significance of system matrix elements

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

a) If $D = 0 \rightarrow \alpha_f = Cy_0$ independent of α_0

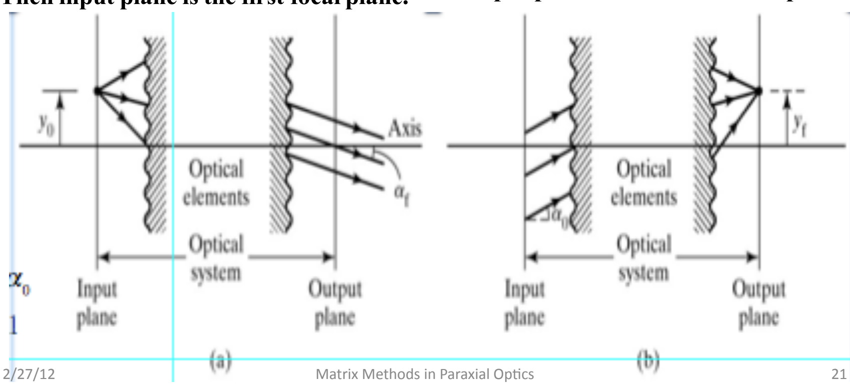
All the rays leavoing the input plane will have the same angle at the output plane.

Then input plane is the first focal plane.

b) If A = 0 $\rightarrow y_f = B\alpha_0$

means y_f is independent of y_0 that means all the rays departing input plane have the same height at the output plane.

Output plane is the second focal plane.



Significance of system matrix elements

c) If $B = 0 \rightarrow y_f = Ay_0$ All the points leaving the input plane at hight y_0 will arrive the output plane at height y_f.

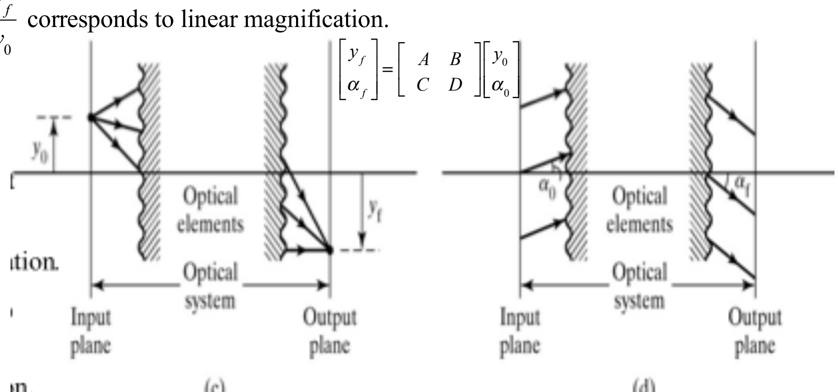
Output plane is image of the input plane.

$A = \frac{y_f}{y_0}$ corresponds to linear magnification.

d) If $C = 0 \rightarrow a_f = D\alpha_0$ independent of y_0

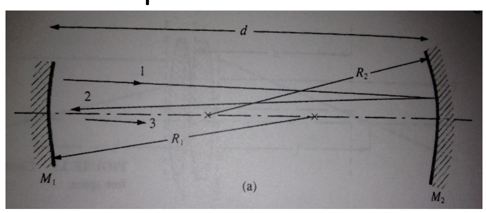
Input rays of all in one direction will produce output rays all in another direction.

This is called telescopic system.



Optical cavities

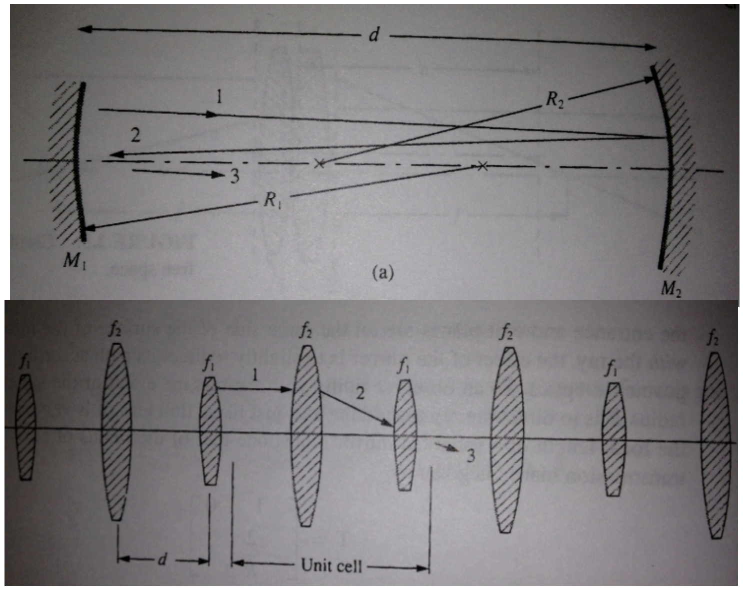
- Optical cavities are composed of two face-to-face reflectors with space in between. They are:
  - **Stable** if the light bounces back an forth between the reflectors without escaping from the cavity
  - **Unstable** if the light "walks off" from the cavity after few round trips
  - **Conditionally stable** if the stability depends on very tight alignment requirements



Unfolding the optical systms

- Any reflection in an optical system can be unfolded.
- Unfolding converts a **reflection problem to a translation and refraction** problem so we don't have to deal with the constant change of direction of propagation of light.

Equivalent lens waveguide



RTM of a cavity

Choose the unit cell & construct the RTM of the unit cell

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix}}_{\text{Fourth}} \underbrace{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}}_{\text{Third}} \underbrace{\begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix}}_{\text{Second}} \underbrace{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}}_{\text{First}} \underbrace{\begin{bmatrix} r_s \\ r'_s \end{bmatrix}}_{\text{Incident}}$$

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \begin{bmatrix} \left(1-\frac{d}{f_2}\right) & d + d\left(1-\frac{d}{f_2}\right) \\ -\frac{1}{f_1} - \frac{1}{f_2}\left(1-\frac{d}{f_1}\right) & \left(1-\frac{d}{f_1}\right)\left(1-\frac{d}{f_2}\right) - \frac{d}{f_1} \end{bmatrix} \begin{bmatrix} r_s \\ r'_s \end{bmatrix}$$

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_s \\ r'_s \end{bmatrix}$$

Finding the difference equation for the rays succeeding through a unit cell

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_s \\ r'_s \end{bmatrix}$$

$$r_{s+1} = Ar_s + Br'_s \rightarrow r'_s = \frac{1}{B}\left(r_{s+1} - Ar_s\right)$$

$$\left.\begin{matrix} r'_{s+1} = Cr_s + Dr'_s = Cr_s + \frac{D}{B}\left(r_{s+1} - Ar_s\right) \\ \text{one step further} \rightarrow r'_{s+1} = \frac{1}{B}\left(r_{s+2} - Ar_{s+1}\right) \end{matrix}\right\} \rightarrow$$

$$\frac{1}{B}\left(r_{s+2} - Ar_{s+1}\right) = Cr_s + \frac{D}{B}\left(r_{s+1} - Ar_s\right) \text{ and with } AD - CB = 1$$

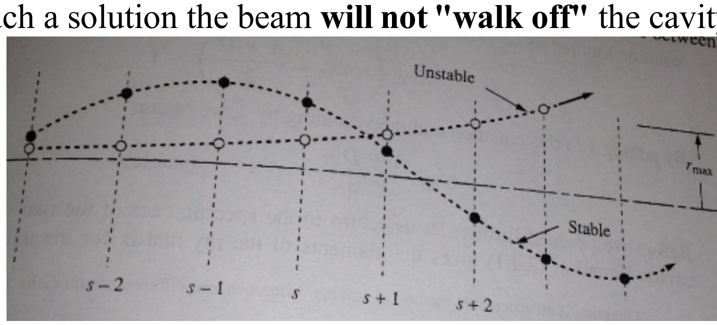
$$r_{s+2} - 2\left(\frac{A+D}{2}\right)r_{s+1} + r_s = 0$$

Desired solutions

Second order difference equation relating the successive heights of the rays in the cavity $\rightarrow r_{s+2} - 2\left(\frac{A+D}{2}\right)r_{s+1} + r_s = 0$

For the good solutions magnitude of r stays below a maximum **for any s** or: $r < r_{\max}$

For such a solution the beam **will not "walk off"** the cavity.



Desired solutions

$$r_{s+2} - 2\left(\frac{A+D}{2}\right) r_{s+1} + r_s = 0; \quad \text{A trial solution: } r_s = r_0 \left(e^{j\theta}\right)^s = r_0 e^{js\theta}$$

where r_0 is specified by the initial conditions

$$r_0 e^{j\theta(s+2)} - 2\left(\frac{A+D}{2}\right) r_0 e^{j\theta(s+1)} + r_0 e^{j\theta s} = 0$$

$$\underbrace{r_0 e^{js\theta}}_{\text{A non-zero term in general}} \left[e^{j2\theta} - 2\left(\frac{A+D}{2}\right) e^{j\theta} + 1 \right] = 0 \rightarrow \underbrace{\left(e^{j\theta}\right)^2 - 2\left(\frac{A+D}{2}\right) e^{j\theta} + 1 = 0}_{\text{A quadratic equation}}$$

$$e^{j\theta} = \frac{A+D}{2} \pm \left[\left(\frac{A+D}{2}\right)^2 - 1 \right]^{1/2} = \frac{A+D}{2} \pm j \left[1 - \left(\frac{A+D}{2}\right)^2 \right]^{1/2}$$

we can use these to construct our real solution.

Stability condition

We combine our solutions with their complex conjugate for a real solution:

$$\begin{cases} r_s = r_0 e^{js\theta} + r_0^* e^{-js\theta} \\ r_s = r_{\max} \sin\left(s\theta + \alpha\right) \leftarrow \text{A real solution} \end{cases}$$

But we have to make sure $e^{j\theta}$ is complex so that summing it with its complex-conjugate will give us a real number. That also means θ is real which means:

$$e^{j\theta} = \cos\theta \pm j\sin\theta = \underbrace{\frac{A+D}{2}}_{\text{Real}} \pm j \underbrace{\left[1 - \left(\frac{A+D}{2}\right)^2 \right]^{1/2}}_{\text{Real}} \rightarrow$$

$$-1 \le \left(\cos\theta = \frac{A+D}{2} \right) \le 1 \quad \text{or} \quad -1+1 \le \left(\frac{A+D}{2} + 1 \right) \le 1+1 \rightarrow$$

Stability condition for cavity $\rightarrow \; 0 \le \dfrac{A+D+2}{4} \le 1$

Stability condition for spherical cavity

The stability condition for a cavity: $0 \le \left(\dfrac{A + D + 2}{4} \right) \le 1$

Recall our cavity with spherical mirrors:

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \begin{bmatrix} \left(1 - \dfrac{d}{f_2}\right) & d + d\left(1 - \dfrac{d}{f_2}\right) \\ -\dfrac{1}{f_1} - \dfrac{1}{f_2}\left(1 - \dfrac{d}{f_1}\right) & \left(1 - \dfrac{d}{f_1}\right)\left(1 - \dfrac{d}{f_2}\right) - \dfrac{d}{f_1} \end{bmatrix} \begin{bmatrix} r_s \\ r'_s \end{bmatrix}$$

Stability condition for a cavity with spherical morrors :

$$\frac{A + D + 2}{4} = \frac{1}{4}\left[1 - \frac{d}{f_2} - \frac{d}{f_1} + \left(1 - \frac{d}{f_1}\right)\left(1 - \frac{d}{f_2}\right) + 2\right] = \underbrace{\left(1 - \frac{d}{2f_1}\right)}_{g_1}\underbrace{\left(1 - \frac{d}{2f_2}\right)}_{g_2}$$

$$0 \le g_1 g_2 \le 1;\ g_1 = 1 - \frac{d}{R_1}\ \&\ g_2 = 1 - \frac{d}{R_2}\ \&\ R_1 = 2f_1\ \&\ R_2 = 2f_2$$

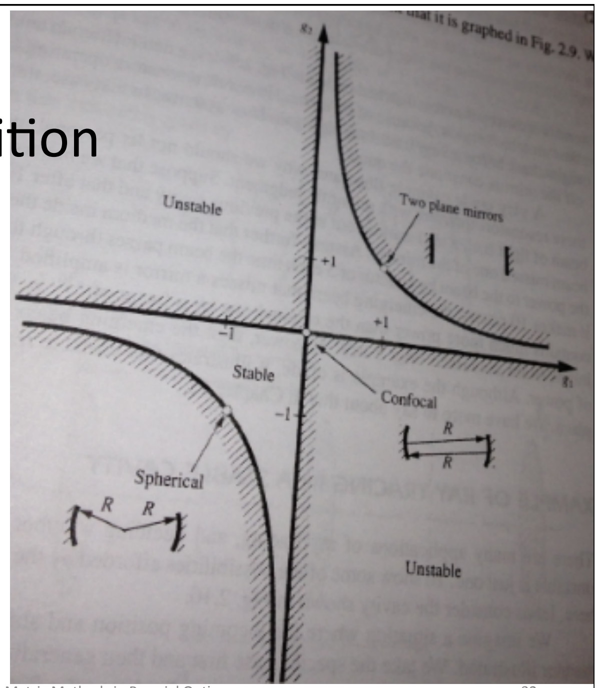
Graph of the stability condition

$0 \le g_1 g_2 \le 1$

$g_1 = 1 - \dfrac{d}{R_1}$

$g_2 = 1 - \dfrac{d}{R_2}$

A very important and useful equation and graph for determining stability of a cavity by a **glance**.

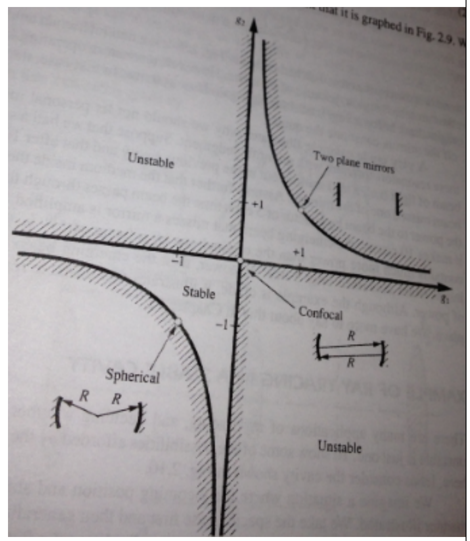


Unstable region

The unstable region is described by

$$\left(\frac{A+D}{2}\right)^2 > 1$$

Shown by cross hatched region.
This region is used for design of high-power lasers.
The walked off rays become the output. Chapter 12 talks about these resonators.



Example of ray tracing in stable cavity A repetitive ray path

A cavity with $R_1 = \infty$ and R_2

Initial ray height $= -r_0$; the **initial angle is zero**.

Initial ray hits the spherical mirror, reflection passes through the f_2

Reflection from the flat mirror alng the radius hits the spherical mirror and back at itself forever.

Is this cavity stable?

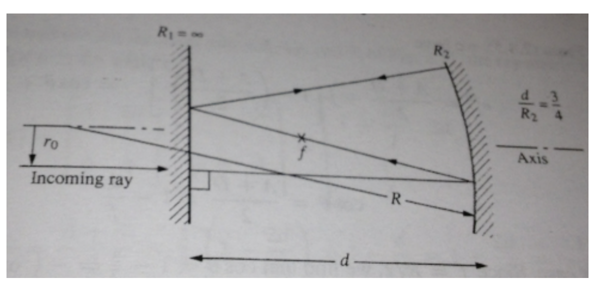
$$g_1 = 1 - \frac{d}{R_1} = 1$$

$$g_2 = 1 - \frac{d}{R_2} = 1 - \frac{3}{4} = \frac{1}{4}$$

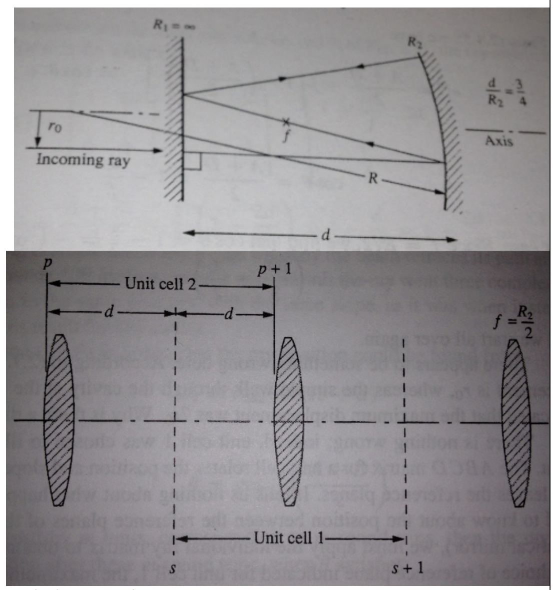
$$g_1 g_2 = 1/4 < 1$$

Yes

What is the r_{max} ? $2r_0$ (show it)



Lens-waveguide equivalent



We have many choices for the unit cell. Two of them are shown in the figure

Solution for repetitive ray path cavity

$$T_1 = \underbrace{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}}_{\text{Third}} \underbrace{\begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix}}_{\text{Second}} \underbrace{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}}_{\text{First}} \to T_1 = \underbrace{\begin{bmatrix} 1-d/f & d+d(1-d/f) \\ -1/f & 1-d/f \end{bmatrix}}_{\text{RTM of the first choice unit cell}}$$

The two form of the solution: $\begin{cases} r_s = r_0 e^{js\theta} + r_0^* e^{-js\theta} \\ r_s = r_{\max}\sin(s\theta+\alpha) \end{cases}$

$$e^{j\theta} = \frac{A+D}{2} + j\left[1-\left(\frac{A+D}{2}\right)^2\right]^{1/2} = \cos\theta \pm j\sin\theta \text{ with } \frac{d}{R_2} = \frac{3}{4}$$

$$\cos\theta = \frac{A+D}{2} = 1-\frac{d}{f} = 1-\frac{d}{R_2/2} = 1-2\left(\frac{3}{4}\right) = -\frac{1}{2} \to \theta = \frac{2\pi}{3} = 120^0$$

using the initial conditions $r = -r_0 \xrightarrow{\text{for s=0 and } r_{\max}=r_0} r_s = r_0 \sin\left(\frac{2\pi}{3}s - \frac{\pi}{2}\right)$

$$s=0 \to r_s = -r_0; \quad s=1 \to r_s = \frac{r_0}{2}; \quad s=2 \to r_s = \frac{r_0}{2}; \quad s=3 \to r_s = -r_0$$

back to the original value of the start of the round trip.

Solution for repetitive ray path cavity

$$T_1 = \underbrace{\begin{bmatrix} 1 & 2d \\ 0 & 1 \end{bmatrix}}_{\text{Second}} \underbrace{\begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix}}_{\text{First}} \to T_1 = \underbrace{\begin{bmatrix} 1-2d/f & 2d \\ -1/f & 1 \end{bmatrix}}_{\text{RTM of the second choice unit cell}}$$

The two form of the solution: $\begin{cases} r_p = r_0 e^{jp\theta} + r_0^* e^{-jp\theta} \\ r_p = r_{\max}\sin(p\theta+\alpha) \end{cases}$

$$e^{j\theta} = \frac{A+D}{2} + j\left[1-\left(\frac{A+D}{2}\right)^2\right]^{1/2} = \cos\theta \pm j\sin\theta \text{ with } \frac{d}{R_2} = \frac{3}{4}$$

$$\cos\theta = \frac{A+D}{2} = \frac{1}{2}\left(2-\frac{2d}{f}\right) = 1-\frac{d}{R_2/2} = 1-2\left(\frac{3}{4}\right) = -\frac{1}{2} \to \theta = \frac{2\pi}{3}$$

using the initial conditions $r = r_0 \xrightarrow{\text{Solution}} r_p = 2r_0\sin\left(\frac{2\pi}{3}p-\frac{\pi}{6}\right)$

$p=0 \to r_p = r_0;\quad p=1 \to r_p = 2r_0;\quad p=2 \to r_p = -r_0;$

back to the original value of the start of the round trip. This choice of the unit cell **does not give any information about the ray at the flat mirror.**

How many roundtrips to get back to the original position?

For any cavity we can find the number of round trips it takes to get the beam to its original position.

$$r_s = r_{\max}\sin\left(s\underbrace{\theta}_{\text{phase gain per round trip}}+\alpha\right) \text{ where } \theta=\cos^{-1}\left(\frac{A+D}{2}\right)$$

Assume s increases by m units to get back to the original position.

Total phase gain has to be integer of 2π

$m\theta = 2\pi n$ and to guarantee $\theta<\pi$ we require $m > 2n$

For our solution: $r_s = r_0\sin\left(\frac{2\pi}{3}s-\frac{\pi}{2}\right) \to m\frac{2\pi}{3} = 2\pi n \to m = 3n$

$n=1 \to m=3$ and $m > 2n$ holds so **after 3 round trips**

For The other solution m is the same since θ is the same.

Initial conditions: Stable cavities

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_s \\ r'_s \end{bmatrix} \rightarrow \underbrace{-1 \leq \left(\cos\theta = \frac{A+D}{2} \right) \leq 1}_{\text{Stability condition}} \rightarrow \theta = \cos^{-1}\left(\frac{A+D}{2} \right)$$

Solution: $r_s = r_{max} \sin(s\theta + \alpha)$; Initial $(s = 0)$ Ray position: $r_0 = a$; Ray slope: $r'_0 = m$

$s = 0 \rightarrow a = |r_{max}| \sin(\alpha)$

$s = 1 \rightarrow r_1 = |r_{max}| \sin(\theta + \alpha) = |r_{max}| \sin\theta \cos\alpha + |r_{max}| \cos\theta \sin\alpha$

Also: $r_1 = Ar_0 + Br'_0 = Aa + Bm$

Then $r_1 = Aa + Bm = |r_{max}| \sin\theta \cos\alpha + \frac{A+D}{2} a$

$$|r_{max}| \cos\alpha = \frac{1}{\sin\theta} \left[a \frac{A-D}{2} + Bm \right] = \frac{a}{\sin\alpha} \cos\alpha \rightarrow \tan\alpha = \frac{a \sin\theta}{a \frac{A-D}{2} + Bm}$$

The phase angle for the stable solution is: $\alpha = \tan^{-1} \left[\frac{a \left(1 - \left(\frac{A+D}{2} \right)^2 \right)^{1/2}}{a \frac{A-D}{2} + Bm} \right]$ and $r_{max} = \frac{a}{\sin\alpha}$

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Initial conditions: Stable cavities Example

For the previous problem:

$a = \text{-}r_0$ & $m = 0$

$$\alpha = \tan^{-1} \left[\frac{a \left(1 - \left(\frac{A+D}{2} \right)^2 \right)^{1/2}}{a \frac{A-D}{2} + Bm} \right] = \tan^{-1} \left[\frac{-r_o \left(1 - \left(\frac{2\pi}{3} \right)^2 \right)^{1/2}}{-r_0(0) + B(0)} \right] = \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

$$r_{max} = \frac{a}{\sin\alpha} = \frac{-r_0}{\sin(-\pi/2)} = \frac{-r_0}{-1} = r_0$$

For the second solution

$a = -r_0$ & $m = \textit{need to know it}$

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Initial conditions: unstable cavities

Back to the difference equation: $r_{s+2} - 2\frac{A+D}{2} r_{s+1} + r_s = 0$; let $r_s = r_0 (F)^s$

$$rF^s\left[F^2 - 2\left(\frac{A+D}{2}\right)F + 1\right] = 0 \xrightarrow[\text{for quadratic}]{\text{Solutions}} \begin{cases} F_1 = \frac{A+D}{2} + \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{1/2} \\ F_2 = \frac{A+D}{2} - \left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{1/2} \end{cases}$$

$r_s = r_a (F_1)^s + r_b (F_2)^s$

For unstable cavities $\left|(A+D)/2\right| > 1$ that means one of the solutions $F_1 > 1$ or $F_2 > 1$

After few roundtrips ray's position is is mostly increasing exponentially due to the larger solution and gets further away from the axis: $r_s \sim r_0 (F_>)^s$

For $s = 0 \rightarrow r_0 = a = r_a + r_b$ & For $s = 1 \rightarrow r_1 = aA + Bm = r_a F_1 + r_b F_2$

We can find the r_a and r_b by solving $\begin{cases} r_a = \frac{1}{F_1 + F_2}\left[a(F_1 - A) - Bm\right] \\ r_b = \frac{1}{F_2 - F_1}\left[a(F_2 - A) - B_m\right] \end{cases}$

Astigmatism

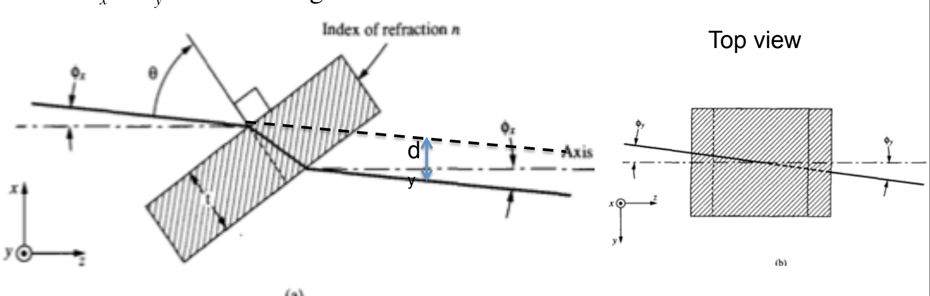
When a material body is placed in the path of a ray and is tilted we must account for the change in the optical path in two orthogonal directions

Optical paths traversed by the two rays through a **Brewester angle** window is:

$$d_y = t\frac{(n^2+1)}{n^2} \ \& \ d_x = t\frac{(n^2+1)^{1/2}}{n^4}$$

For the Brewester abgle: $\tan\theta_B = n$

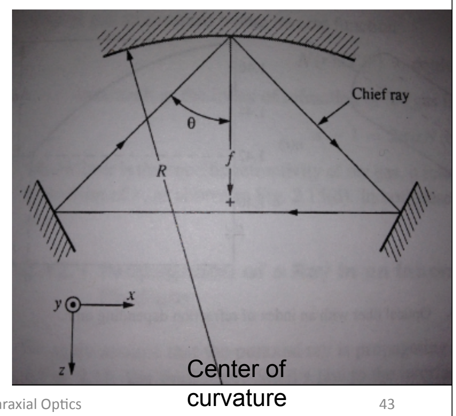
Since $d_x \neq d_y$ we have astigmatism.



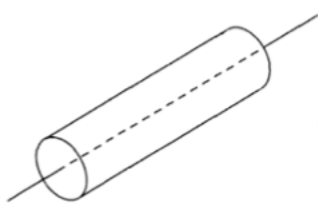
Astigmatism and ring lasers

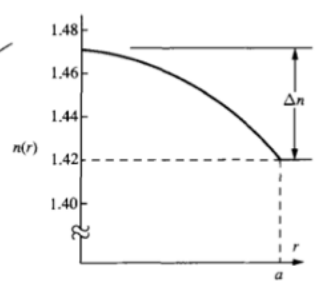
- Astigmatism plays a critical role in dye-laser cavities
- It leads to elliptical beams.
- Focal lengths in xy and xz planes is different.

$$f_x = f\cos\theta$$
$$f_y = f/\cos\theta$$

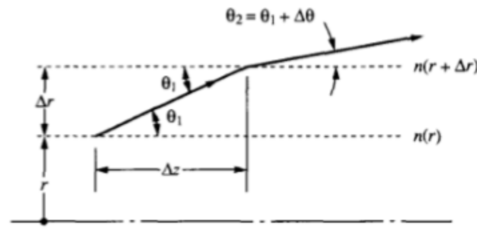


Continuous lens-like media





Propagation of ray in an inhomogeneous medium



Ray matrix for continuous lens