Superposition of waves
Superposition of waves is the common conceptual basis for some optical phenomena such as:
- Polarization
- Interference
- Diffraction

What happens when two or more waves overlap in some region of space.

How the specific properties of each wave affects the ultimate form of the composite disturbance?

Can we recover the ingredients of a complex disturbance?
Linearity and superposition principle

The scalar 3D wave equation \( \frac{\partial^2 \psi(r,t)}{\partial r^2} = \frac{1}{V^2} \frac{\partial^2 \psi(r,t)}{\partial t^2} \) is a linear differential equation (all derivatives appear in first power). So any linear combination of its solutions \( \psi(r,t) = \sum_{i=1}^{n} C_i \psi_i(r,t) \) is a solution.

**Superposition principle**: resultant disturbance at any point in a medium is the algebraic sum of the separate constituent waves.

We focus only on **linear systems and scalar functions** for now. At high intensity limits most systems are nonlinear.

Example: power of a typical focused laser beam \( \approx 10^{10} \) V/cm compared to sun light on earth \( \approx 10 \) V/cm. Electric field of the laser beam triggers nonlinear phenomena.
Superposition of two waves

Two light rays with same frequency meet at point p traveled by $x_1$ and $x_2$

$$E_1 = E_{01} \sin[\omega t - (kx_1 + \epsilon_1)] = E_{01} \sin[\omega t + \alpha_1]$$

$$E_2 = E_{02} \sin[\omega t - (kx_2 + \epsilon_2)] = E_{02} \sin[\omega t + \alpha_2]$$

Where $\alpha_1 = -(kx_1 + \epsilon_1)$ and $\alpha_2 = -(kx_2 + \epsilon_2)$

Magnitude of the composite wave is sum of the magnitudes at a point in space & time or: $E = E_1 + E_2 = E_0 \sin(\omega t + \alpha)$ where

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos(\alpha_2 - \alpha_1)$$ and $\tan \alpha = \frac{E_{01} \sin \alpha_1 + E_{02} \sin \alpha_2}{E_{01} \cos \alpha_1 + E_{02} \cos \alpha_2}$

The resulting wave has same frequency but different amplitude and phase.

$2E_{01}E_{02} \cos(\alpha_2 - \alpha_1)$ is the interference term

$\delta \equiv \alpha_2 - \alpha_1$ is the phase difference.
Phase difference and interference

\[ \delta = \alpha_2 - \alpha_1 = (kx_1 + \varepsilon_1) - (kx_2 + \varepsilon_2) = \frac{2\pi}{\lambda}(x_1 - x_2) + (\varepsilon_1 - \varepsilon_2) = \delta_1 + \delta_2 \]

Total phase difference between the two waves has two different origins.

a) \( \delta_2 = (\varepsilon_1 - \varepsilon_2) \) phase difference due to the initial phase of the waves. Waves with constant initial phase difference are said to be coherent.

b) \( \delta_1 = \frac{2\pi}{\lambda_0} n(x_1 - x_2) = k_0 \Lambda \) is phase difference due to the Optical Path Difference or OPD \( \Lambda = n(x_1 - x_2) \)

Waves in-phase: \( \delta \equiv \alpha_2 - \alpha_1 = 0, \pm 2\pi, \pm 4\pi, \ldots \) then \( E_0 \) is maximum

Waves out of phase: \( \delta \equiv \alpha_2 - \alpha_1 = \pm \pi, \pm 3\pi, \ldots \) then \( E_0 \) is minimum

Waves in-phase interfere constructively \( E = E_{\text{max}} = (E_{01} + E_{02})^2 \)

Waves out of phase interfere destructively \( E = E_{\text{min}} = (E_{01} - E_{02})^2 \)

If \( E_{01} = E_{02} \) then \( E_{\text{max}} = (2E_{01})^2 \) and \( E_{\text{min}} = 0 \)

\[ \frac{\Lambda}{\lambda_0} = \frac{x_1 - x_2}{\lambda} \] is the number of waves in the medium
Addition of Two waves with same frequency
Two waves with path difference

For two waves with no initial phase difference \((\varepsilon_1 = \varepsilon_2 = 0)\) but a path difference of \(\Delta x\) we have:

\[
E_1 = E_{01} \sin(\omega t - k(x + \Delta x)) = E_{01} \sin(\omega t + \alpha_1)
\]

\[
E_2 = E_{02} \sin(\omega t - kx) = E_{02} \sin(\omega t + \alpha_2)
\]

\[
\alpha_2 - \alpha_1 = k\Delta x
\]

The resulting wave is

\[
E = 2E_0 \cos \left( \frac{k\Delta x}{2} \right) \sin \left[ \omega t - k \left( x + \frac{\Delta x}{2} \right) \right]
\]

Constructive interference: if \(\Delta x \ll \lambda\), or \(\Delta x/\lambda \approx \pm 2m\) then the resulting amplitude is \(\sim 2E_0\)

Destructive interference: \(\Delta x \approx \pm m\lambda / 2\) then \(E \approx 0\)
Exercise

RV2-1) Plot \( E_1, \ E_2, \ E_1 + E_2, \) and \((E_1 + E_2)^2\) for the following two sinusoidal waves for \(0 < x < 5\lambda\) with \(\lambda = 500\ nm:\n\ E_1 = E_0_1 \sin(\omega t - (kx + \varepsilon_1))\) and \(E_2 = E_0_2 \sin(\omega t - (kx + \varepsilon_2))\)

a) same frequency, \(E_0_1 = E_0_2 = 2\), zero initial phase, both forward.
b) same frequency, \(E_0_1 = E_0_2 = 2\), \(\varepsilon_1 = 0, \ \varepsilon_2 = \pi\), both forward.
c) same frequency, \(E_0_1 = E_0_2 = 2\), \(\varepsilon_1 = 0, \ \varepsilon_2 = \pi / 2\), both forward.
d) same frequency, \(E_0_1 = E_0_2 = 2\), \(\varepsilon_1 = 0, \ \varepsilon_2 = \pi\), \(E_1\) forward, \(E_2\) backward.
e) same frequency, \(E_0_2 = 2E_0_1 = 2\), \(\varepsilon_1 = 0, \ \varepsilon_2 = 0\), both forward.
f) same frequency, \(E_0_2 = 2E_0_1 = 2\), \(\varepsilon_1 = 0, \ \varepsilon_2 = \pi\), both forward.
g) Compare the results of direct superposition with the formula derived in text for case a (slide 4). (Notice the difference between \(\text{atan}\) and \(\text{atan}^2\) functions in MATLAB)
Phasors and complex number representation

- Each harmonic function is shown as a rotating vector (phasor)
  - projection of the phasor on the x axis is the **instantaneous value of the function**, 
  - length of the phasor is the maximum amplitude 
  - angle of the phasor with the positive x direction is the **phase of the wave**.

\[
E = E_0 e^{i(\omega t + \alpha)}
\]

\[
E(t) = E_0 \cos(\omega t + \alpha_1)
\]

\[
E(t) = E_0 \sin(\omega t + \alpha_1)
\]
Superposition using phasors

\[ E_1(t) = E \cos(\omega t + \phi) \quad E_2(t) = E \cos(\omega t) \]
\[ \vec{E}_p = \vec{E}_1 + \vec{E}_2 \text{ a vector sum of } \vec{E}_1 \text{ and } \vec{E}_2 \]

Magnitude of \( \vec{E}_p \) (from triangonometry)

\[ E_p^2 = E_1^2 + E_2^2 - 2E_1E_2 \cos(\pi - \phi) \]
\[ E_p^2 = E_1^2 + E_2^2 + 2E_1E_2 \cos \phi \]

Using \( 1 + \cos \phi = 2 \cos^2(\phi / 2) \)

\[ E_p^2 = 2E_1^2(1 + \cos \phi) = 4E_1^2 \cos^2(\phi / 2) \]

Amplitude of two wave interference independent of time:

\[ E_p = 2E \left| \cos \frac{\phi}{2} \right| \]
Superposition of many waves

Superposition of any number of coherent harmonic waves with a given frequency, $\omega$ and traveling in the same direction leads to a harmonic wave of that same frequency.

$$E = \sum_{i=1}^{N} E_{0i} \cos(\alpha_i \pm \omega t) = E_0 \cos(\alpha \pm \omega t)$$

$$E_0^2 = \sum_{i=1}^{n} E_{0i}^2 + 2 \sum_{j>i}^{N} E_{0i} E_{0j} \cos(\alpha_i - \alpha_j) \quad \text{and} \quad \tan \alpha = \frac{\sum_{i=1}^{N} E_{0i} \sin \alpha_i}{\sum_{i=1}^{N} E_{0i} \cos \alpha_i}$$

For coherent sources $\alpha_i = \alpha_j$ and $E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2 \sum_{j>i}^{N} E_{0i} E_{0j} = (NE_{0i})^2$

For incoherent sources (random phases) the second term is zero.

Flux density for N emmiters: $(E_0^2)_{\text{incoherent}} = NE_{01}^2; \quad (E_0^2)_{\text{coherent}} = (NE_{0i})^2$
Exercise

RV2-2) Write a MATLAB routine to calculate the amplitude and phase of N harmonic waves (cosine) with same frequencies but varying initial phase and amplitudes. Assume the wavelength is 500 nm and $V = c$

a) The program should read the phase and amplitude of the waves from a file that has two columns and N rows. Test the program for the following waves $E_1 = 1, \varepsilon_1 = 0, E_2 = 1, \varepsilon_2 = \pi/4$. Once made sure it is working, create a file with the following waves and plot their superposition from 0 to $5\lambda$.

$E_1 = 1, \varepsilon_1 = 0, E_2 = 1, \varepsilon_2 = 10, E_3 = 2, \varepsilon_3 = 20^0, E_4 = 3$, 
$\varepsilon_4 = 30^0, E_5 = 2, \varepsilon_5 = 40^0, E_6 = 1, \varepsilon_6 = 50^0, E_7 = 1, \varepsilon_7 = 60^0$

b) Next run the program for N=100 and $\varepsilon_i = \varepsilon_1 + \frac{i}{100} \pi$, where $\varepsilon_1 = \frac{\pi}{2}$ and $E_i = 2$. This time create the phases and amplitudes inside the routine and don't read from a file.
Addition of waves: different frequencies I

Mathematics behind light modulation and light as a carrier of information. Two propagating waves are superimposed

\[ E_1 = E_{01} \cos(k_1 x - \omega_1 t) \]
\[ E_2 = E_{01} \cos(k_2 x - \omega_2 t) \]

\( k_1 > k_2 \) and \( \omega_1 > \omega_2 \) with equal amplitudes and zero initial phases

\[ E = E_1 + E_2 = E_{01}[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)] \]

using \( \cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) \)

\[ E = 2E_{01} \cos \frac{1}{2}[(k_1 + k_2) x - (\omega_1 + \omega_2) t] \times \cos \frac{1}{2}[(k_1 - k_2) x - (\omega_1 - \omega_2) t] \]

Need to simplify this
Addition of waves: different frequencies II

\[ E = 2E_0 \cos[k_m x - \omega_m t] \times \cos[\bar{k}x - \bar{\omega}t] \]

with the following definitions

Average angular frequency \( \bar{\omega} = \frac{1}{2} (\omega_1 + \omega_2) \)

Average propagation number \( \bar{k} = \frac{1}{2} (k_1 + k_2) \)

Modulation angular frequency \( \omega_m = \frac{1}{2} (\omega_1 - \omega_2) \)

Modulation propagation number \( k_m = \frac{1}{2} (k_1 - k_2) \)

Time-varying modulation amplitude \( E_0(x,t) = 2E_0 \cos[k_m x - \omega_m t] \)

Superimposed wavefunction: \( E = E_0(x,t) \cos[\bar{k}x - \bar{\omega}t] \)

For large \( \omega \) if \( \omega_1 \approx \omega_2 \) then \( \bar{\omega} \gg \omega_m \) we will have a slowly varying amplitude with a rapidly oscillating wave.
Irradiance of two superimposed waves with different frequencies

\[ E_0^2(x, t) = 4E_0^2 \cos^2[k_m x - \omega_m t] = 2E_0^2 \left[1 + \cos(2k_m x - 2\omega_m t) \right] \]

Beat frequency \( \text{Beat frequency} = 2\omega_m = \omega_1 - \omega_2 \) or oscillation frequency of the \( E_0^2(x, t) \)

Amplitude, \( E_0 \), oscillates at \( \omega_m \), the modulation frequency

Irradiance, \( E_0^2 \), varies at \( 2\omega_m \), twice the modulation frequency

Two waves with different amplitudes produce beats with less contrast.
Beats

Superposition of two waves

\[ \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \]

\[ 4E_0^2 \]

\[ 2E_0^1 \]

Distance

Amplitude

\[ \lambda_1 \]

\[ \lambda_2 \]

\[ \lambda_m \]

Distance \( x \times 10^{-6} \)
Group velocity

In nondispersive media velocity of a wave is independent of its frequency.

For a single frequency wave there is one velocity and that is $V_{phase} = \frac{\omega}{k}$.

When a wave is composed of different frequency elements, the resulting disturbance will travel at a different velocity than phase velocity of its components.

$$E = 2E_0 \cos[k_m x - \omega_m t] \times \cos[-k x - \omega t]$$

$V_{phase} = \frac{\omega}{k}$ velocity of a constant phase point on the high frequency wave

$$V_{group} = \frac{\omega_m}{k_m} = \left(\frac{d\omega}{dk}\right)_{\omega}$$ velocity of the modulation envelope

$V_g$ may be smaller, equal, or larger than $v_p$.

To calculate the $V_p$ and $V_g$ we need the dispersion relation $\omega = \omega(k)$. 
Finite waves

- Finite wave: any wave starts and ends in a certain time interval
- Any finite wave can be viewed as a really long pulse
- Any pulse is a result of superposition of numerous different frequency harmonic waves called **Fourier components**.
- **Wave packet** is a localized pulse that is composed of many waves that cancel each other everywhere else but at a certain interval in space.
- We need to study Fourier Analysis to understand actual waves, pulses, and wave packets.
- Range of $k_m$ is proportional to the wave packet width
- Since each component of the wave packet has different phase velocity in the medium, through the relationship $V_p=\omega/k$, $k$ of the components change in the dispersive media.
- As a result $k_m$ of the modulation disturbance changes and consequently group velocity changes. This results change of the width of the wave packet.